THE ELEMENTS OF EUCLID,

WITH

DISSERTATIONS,

INTENDED

To assist and encourage a critical examination of these Elements, as the most effectual means of establishing a juster taste upon mathematical subjects, than that which at present prevails.

VOL. I.

BY JAMES WILLIAMSON, M.A.
FELLOW OF HERTFORD COLLEGE.

Sed nil dulcius est, bene quam munita tenere
Edita doctrina Sapientum templa serena. LUCR.

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M DCC LXXXI.

TO

RICHARD BURKE, Esq.

DEAR SIR,

IT would be a very entertaining ſpeculation, and well calculated to ſhew the importance of geometry, to enquire into the probable ſtate of human affairs at preſent, upon a ſuppoſition that the mathematical ſciences had been generally known and cultivated in the early ages of the world.

If they had only underſtood how to make a *map* or an *almanack*, a great deal of rational curioſity would have been gratified, which we now endeavour in vain to ſatisfy. For mankind have ſlept on, from age to age, without taking notice of time or occurences; and as to what concerns *geography* they have been equally remiſs. The human race have migrated from one country to another, until they have covered the face of the whole Earth, without gaining the leaſt information by their travels; having rather ſtrayed like cattle, than changed their habitation like rational creatures.

However the happy effects of a proper cultivation of this ſcience, are not confined to *maps* and *almanaſſs*: for ck
it is by the judicious application of mathematical knowledge, that human nature, notwithſtanding its frailty, is rendered ſuperiour to every obſtacle, for our efforts keep pace with all our rational imaginations: and thus

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it rises greatly above all its imperfections, acquiring a dignity which must astonish every one, when he compares what has been done, with the weakness and imperfection of any single individual. And what this might have produced through a length of ages, operating in every quarter of the globe is not so easily to be conceived.

It has indeed been alledged, that the improvements ascribed to the cultivation of this science, have been often the effect of chance, and not produced by any rational scheme of improvement, conducted upon scientific principles. But the contrary appears from undoubted matter of fact; because we every where find that discoveries are made among the most enlightened nations, and never among barbarians. Events, no doubt, will happen, and natural appearances will keep their regular time; but the wild beasts of the field are as likely to make a proper use of them, as illiterate savages. And the little attention paid by ignorant men to those very circumstances which inform and enlarge the understanding, shew us better than any thing else, what human nature can rise and fall to.

Both you and I are sufficiently sensible of the great utility of mathematics in all the affairs of life; and that those whom it concerns are, to a certain degree at least, sufficiently attentive to it. And we only used to regret that the world is ignorant, how successfully this science might be employed for making us rational creatures.

It has often been the subject of our conversation and surprize to run over the various methods contrived for making mathematicians, without imposing upon them the necessity of being rational creatures. And such discourses have generally ended in your urging me to the execution of this plan; the first part of which I here beg leave

leave to present to you. I have had sufficient marks of your friendship to be sensible that you will be much more sollicitous about its success than I am myself: and from your partiality to me, I am also persuaded you will think that I have not done it justice in the execution.

It is true I could have improved the style very much; but it seems to answer my purpose better in its present form: for I write not to make people read, but to make them think. I have also affected a familiarity of phrase, to engage the attention of the reader by expressing geometrical ideas in common language.

But whatever your opinion of the work may be, I beg of you to accept of this address, as a testimony of the high esteem which I have for your character and abilities. It is with the greatest pleasure that I recollect the share which I have had in your education, which was carried on, not in the person of master and scholar; but rather in the character of two friends who had taken a somewhat different view of the same subject.

I am, DEAR SIR,

with the greatest respect,

Your most obedient

humble servant,

HERTFORD COLLEGE:
March 27. 1781.

JAMES WILLIAMSON.

DISSERTATION I.

IT is my intention in this dissertation, after saying something concerning the use of commentaries, to conduct the learner, step by step, to a solid, rational and extensive view of the principles of geometry; by explaining their original and improvement through several different gradations, until they put on a scientific form; reducing what I have to say under distinct heads, for the benefit of the ignorant and thoughtless reader.

CHAP. I.

Of the necessity and use of commentaries.

THE dulness of commentators is a subject of such general complaint, that it may be proper to inquire how it comes to pass that books are not written in such a manner as to make a commentary useless or impertinent: and this inquiry may be proposed with the greater propriety, as the labours of commentators, besides their dulness, seldom or never answer the expectations of the public in other respects; which they no doubt are apt to imagine the author himself could have fully gratified, by condescending a little to the weakness of his readers, and rescued his works out of such clumsy hands.

What apology therefore shall I be able to make for loading with a commentary the most perfect book in the world! But though it may not be agreeable to that general delicacy which an author is bound to preserve towards his readers; yet it may nevertheless be proper to inform them, that they seem to be rather partial to themselves upon the present question; never considering how much of

the blame ought in all conſcience to be laid at their own door: for though it may be a ſecret to a great many, yet it is an undoubted truth that the moſt perfect writer requires that his work ſhould be put into the hands of thoſe who can read. And my intention in this edition is not to correct my author, but to ſupply a defect which it could not have been very conſiſtent with his plan to remedy. For he has written his book expreſſly upon the ſuppoſition that his reader was endued with the faculty of attention: and as this is a diſpoſition of mind with which the book is but rarely taken up, though it will always be laid down with it, or elſe it has been uſed to little purpoſe: a few ſeaſonable warnings therefore, to rouſe the attention of the indolent reader, may be given with great propriety, and without bringing any reflexion upon the character of the author, who in point of perſpicuity, excells every one.

But I will even go farther and venture to affirm that an author, who writes upon ſubjects of ſcience, may find it often by no means convenient to deliver himſelf in ſuch a manner as to be always intelligible even to thoſe whom he would wiſh to have for readers. Becauſe authors are confined to a particular method of arrangement, if it be their intention to deliver opinions or diſcoveries in a ſyſtematic manner. And although there has been a ſenſeleſs and inceſſant clamour, for almoſt two centuries, againſt this method; yet it ſeems to be the only way of keeping knowledge within ſuch limits, as to be by any means manageable by the human mind.

It is true indeed that our progreſs in acquiring knowledge, when left to ourſelves, and our own experience, is directly contrary to this, being firſt condemned to an examination of particulars. And even when we take up with the ſyſtems of others, it is requiſite that we have laid in a ſufficient ſtock of materials to enable us readily to comprehend any very general doctrine. For no general rule, or law, or theorem, or what you pleaſe to call it, is by any means to be underſtood, unleſs we have particular inſtances ready to apply, as occaſion requires: always however excepting a *genius* of the kind given to *Hudibras* in the following lines, containing the meaning and importance of many volumes.

His notions fitted things ſo well,
That which was which he could not tell.

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The fault therefore is not in the ſyſtems, but ariſes from the general incapacity which mankind ſeem to labour under, for judging of the merits of ſuch as have been offered to their conſideration; and as there are quacks of all denominations, ever lying in wait to take advantage of the ſimplicity of the multitude; the world by this means has been over-run with counterfeits.

To have a genius for any ſcience ſeems to me to imply a readineſs at finding out particular inſtances to apply to any general rule. It is therefore to be ſuſpected that thoſe who are deficient in this kind of invention will by no means find their progreſs in any ſcience, anſwerable to the time which they beſtow upon it: they may commit the rules or the theorems to memory and nevertheleſs be ignorant of their meaning and application. Hence a certain degree of invention becomes neceſſary even for the ready acquiſition of a ſcience; and this perhaps not ſo different from that kind of invention by which the principles were at firſt diſcovered, as many have been apt to imagine.

Hence the great use of [illegible] of [illegible] of practice [illegible]

Now here is the difficulty; a ſcientific book ought to contain knowledge in that compact form; in which every one would chuſe to take a review of it, after he has made himſelf maſter of the ſubject: and not incumbered with all thoſe particular inſtances, which would now ſtand in the way of the readers imagination as much as they aſſiſted it before. For inſtance, who could reliſh the nobleſt of Newton's theorems if it preſented itſelf to his mind encumbered with all thoſe properties, which had led him from common notions to the right underſtanding of ſo ſublime a ſpeculation.

A book therefore, if this reaſoning be juſt, which would be proper for a learner would be fit for nobody elſe: and the greateſt perfection of writing will ſtill leave occaſion and employment for the talents of that kind of commentator, which I profeſs myſelf to be. The ſweepings of the author's ſtudy would furniſh the beſt and moſt authentic materials for works of this kind; and yet humble as the office may ſeem, if I can execute my taſk but nearly up to the idea which I have of it, I ſhall not regret my labour. The reader however will be diſappointed, if he expect to find Euclid either corrected or enlarged, my purpoſe being nothing more than

to conduct the learner to that ſenſe in which the author wiſhed himſelf to be underſtood. An oſtentation of learning is a fault which prevails among many commentators; for they ſeem to be apprehenſive leaſt the world ſhould think they know nothing but their buſineſs: and therefore inſtead of explaining their author, endeavour to perſuade the world that they underſtand the ſubject better than he himſelf did. But I do not write to the ſtrength but to the weakneſs of mankind; and therefore would not chuſe to be conſidered as challenging the whole world to find fault, but only as applying a remedy to a weakneſs which my own experience has found does exiſt.

CHAP. II.

Concerning the Original of the geometrical principles.

QUANTITY of both kinds, extenſion and number, is always forcing itſelf upon us whether we will or not; and muſt therefore leave ſome very fixt and determinate impreſſions upon the minds of the moſt inconſiderate. We cannot ſtretch out our hand without receiving a perception of extenſion; nor open our eyes without ſeeing figured objects, bringing along with them to the mind a conſciouſneſs that they are more in *number* than one. This begets notions, which are by no means peculiar to any ſingle art or profeſſion, but common to all men. The geometrician ſelects the moſt accurate of theſe, and with ſuch materials lays the foundation of his ſcience. Particular circumſtances have rendered ſome notions concerning quantity more invariable than others; or rather ſo fixt that nothing can alter them. As it would diſcover ridiculous affectation and ignorance to pretend to change theſe, which is indeed impoſſible, ſo it is alſo below the dignity of our author to affect to diſguiſe them by any forced or unnatural conſtruction, to make them wear a more philoſophic appearance. But although the twelve common notions which he has ſelected, are to be underſtood according to the vulgar conception of them; yet the learner muſt give them a very particular examination: becauſe it is not ſufficient to have theſe notions, but we muſt alſo have the ready uſe of them

them. And this may be acquired in ſome ſuch manner as the following.

Let the ſtudent provide himſelf with a ruler and compaſſes, and after ſome practice in drawing ſtraight lines, and deſcribing circles; he is next to proceed to the examination of the common notions, as if they were properties of ſtraight lines only, and true of nothing elſe. For without this precaution he will undoubtedly be liable to have the diſtich quoted in the laſt chapter applied to him. And any tincture of the *hudibraſtic genius* diſqualifies a man for this ſcience; and excludes him from a great deal of rational amuſement, to ſay nothing of more ſolid advantages. I ſhall therefore at the porch, not only lend the learner my advice but alſo my aſſiſtance in ſtriping himſelf of thoſe prejudices which would diſgrace his behaviour after he has been admitted into this magnificent temple where all the wonders of the world are diſplayed.

The reader may believe that I never would have introduced this advice with ſo much form and circumſtance, without a firm perſwaſion that it is of the laſt importance. He is therefore immediately to ſet about the work, by deſcribing a circle, not a geometrical but a mechanical circle; and ſuch as any ordinary compaſſes will exhibit; drawing at the ſame time ſeveral ſtraight lines from the center to the circumference. He is next to ſatisfy himſelf of the equality of theſe ſtraight lines, by meaſuring them with his compaſſes: his concluſion will be, that they are equal; and he will find his opinion of their equality grounded upon the firſt common notion; becauſe they are all equal to the ſame length, viz. the diſtance between the extreme points of his compaſſes. But it is carefully to be obſerved that this is not to be made the ſubject of a tranſient reflexion, but of frequent and cloſe meditation; varying the center and radius to the utmoſt limits of the compaſſes; with now and then a thought upon the limited nature and imperfection of the inſtruments.

The ſecond and third of the common notions may be examined by deſcribing two circles with the ſame center, but at different diſtances, and drawing ſtraight lines from the center to the remoteſt circumference; the parts of the ſtraight lines intercepted between the two circumferences are equal; and will illuſtrate the ſecond

ſecond common notion by taking the leſs radius from the greater. And thus we are to proceed untill we have ſatisfied ourſelves that theſe common notions are true at leaſt of ſuch ſtraight lines as we can draw upon a piece of paper.

I beg the reader's pardon for my impertinence; but he is farther to be admoniſhed, that it is not ſufficient to run theſe things over in his own mind; but that he muſt be able to expreſs them to the conviction of a by ſtander; and this will make it neceſſary to diſtinguiſh his lines and circles by the letters of the alphabet.

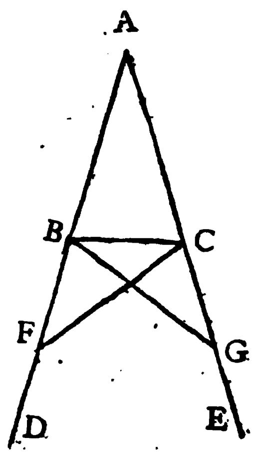
CHAP. III.

The ſame ſubject continued.

SUPPOSING this buſineſs of the ſtraight lines accurately diſcuſſed; the learner is next to ſhut his compaſſes; and then obſerve their progreſs in opening until they take the direction of a ſtraight line: during this operation, he will find the inclination of the legs continually varying: at firſt nothing, then gradually increaſing until it diſappears when the legs become one ſtraight line. This inclination is a quantity, though not a tangible ſubſtance, but this the reader will do well to convince himſelf of; and for this purpoſe he may obſerve that any particular inclination may be equal to another, or the half or the third part of it. But the common notion of this kind of quantity is not ſo regular or determinate as that of a ſtraight line; though it exhibits every poſſible ſhape which it can take in opening the compaſſes as above directed: the reader therefore will be pleaſed to inſtruct himſelf properly in this: and then proceed to examine whether the common notions are not alſo true when applied to this kind of quantity.

And for this purpoſe I would recommend a triangular piece of wood, of the ſhape of a right angled triangle with unequal ſides, being afraid to meddle with circular arches, leaſt we ſhould *conjure up* a prejudice which we might want art afterwards to *lay*. By the aſſiſtance of this triangular piece of wood, make two equal inclinations (or angles) upon paper, taking care to make the lines unequal, to prevent prejudice. After theſe are made, their equality may

may be inferred from the firſt common notion, as each of them will be equal to the inclination of the two ſides of the peice of wood: add to theſe two equal angles, other two equal angles; which may be done by the aſſiſtance of a different corner of the ſame piece of wood; and this will illuſtrate the ſecond and third; according as you conſider one of them as taken away from the whole angle made up of the two; or as added together to make one. But it will be neceſſary previous to this, to acquire a ready and accurate way of expreſſing the different inclinations of lines, (called angles) by the letters of the alphabet. The figure annexed will be a very proper one for practice and the taſk which I would ſet the reader is to tell the number of angles and the different methods of expreſſing them; giving him to underſtand that their number is above fourteen; and that, CAB, CAF, CAD; GAB, GAF, GAD; EAB, EAF, EAD; BAC, BAG, BAE; FAC, FAG, FAE; DAC, DAG, DAE; are only ſo many different ways of expreſſing the ſame angle; nor does this great variety, in the leaſt puzzle or perplex the conceptions of an *adept*. This looks ſo much like a riddle that I think it cannot fail to engage the attention of the curious. But not to truſt entirely to the reader's own ingenuity for unraveling this knotty point; let him obſerve the following hints; the letter at the meeting of the lines, whoſe inclination to one another we want to expreſs, is put in the middle, and it is ſufficient that the other two letters, each expreſs ſome point in each line: thus the inclination of FB to BC is called the angle FBC or CBF: and the inclination of DB to BC is the very ſame with the other, as is obvious, and is called the angle DBC or CBD: the inclination of BC to CE is called the angle BCE or ECB; and the inclination of GC to CB is the ſame with the other and is called the angle GCB or BCG. But farther the angle ABG is made up of two angles viz. ABC, CBG: and the angle ACF is made up of two angles viz. ACB, BCF. And to aſſiſt the reader in applying the ſecond common notion I have made the angle ABG equal



equal to the angle ACF: and I have likewise made the angle CBG equal to the angle BCF; and the conclusion will be that the angle ABC is equal to ACB.

CHAP. IV.

In what manner our common notions begin to take a scientific form.

These are the most important principles to be deduced from Mathematicks.

AFTER the reader has prepared himself according to the directions given in the last two chapters; it will now be proper to take a review of the instruments, which he made use of, for regulating his conceptions: and these, he will find, were very limited, being confined to a few inches. Let him next ask himself, whether he has any reason to suspect, that the conclusions, obtained by the help of these instruments, were equally limited. Nor is this point to be determined rashly, but so as to be still certain that he treads on firm ground: and we may venture to draw this conclusion for him; that, without any great force upon his imagination, he can conceive his instruments to have a double or triple extent without finding the least reason to change his opinions. And by proceeding thus, he will certainly come to this conclusion at last; that although these instruments might be the occasion of his turning his thoughts to this subject, yet his opinions were nevertheless derived from the nature of extension in general; and that they knew no other limits, but such as bounded extension itself: but more particularly, that a circle whose radius is a thousand miles, or the thousand part of an inch, would furnish the same conclusions as one of two or three inches. Here now our opinions, which before were measured by our instruments, begin to put on a different form and display to our imagination the first dawn of science.

If any one should pretend that he had the notions orginally in this very general form to which I have been endeavouring to lead him; I have only to say, unless they were acquired by an examination of particulars, he will find his notions fit every thing so well, that when he comes to apply them to particular instances, he will not be able to tell *which is which*.

The reader is to endeavour next to get something like a scientific notion of an angle, by correcting the vulgar notion of an angle, by

by which is underſtood the corner of any thing. Now this does not ſo much depend upon any ſtretch of the imagination, by which large objects, and ſuch as exceed the experience of our ſenſes, are to be made the ſubject of Contemplation; becauſe the point where the lines meet, together with any point in each of the lines fixes the angle invariably: or in other words, the three points denoted by the three letters of the alphabet, expreſſing the angle, fixes any rectilineal angle: for the angle is not changed by making the lines longer or ſhorter; but only by opening or ſhutting them; conceiving them to turn upon a pin like the two legs of a pair of compaſſes.

But our inſtruments are not only too limited for our conceptions, but are inaccurate in other reſpects. We have a very clear notion of three dimenſions viz. length, breadth and thickneſs: and ſurely without nicely ſeparating and diſtinguiſhing theſe, it is impoſſible to have true and proper conceptions of magnitude. But theſe different dimenſions cannot be repreſented by our inſtruments. For when we attempt to draw a line or even to mark a point; our line and point poſſeſs all the three dimenſions in as great perfection as a cannon ball or the maſt of a ſhip. The human mind, when once made ſenſible of its powers, will never ſuffer its conceptions to be ſo cloged with matter: which has put thoſe who carry their views beyond the vulgar, upon inventing ſome method by which our conceptions may be rendered more rational and conſiſtent; and this is the original of definitions.

CHAP. V.

Of definitions.

OUR author has proceeded with ſingular judgement in laying down his principles; where the common notions are ſufficiently diſtinct and accurate, he has inviolably adhered to them. But when theſe are too incorrect or too indeterminate, he explains the ſenſe in which he would have any particular term be underſtood; and what conception he requires his reader to have of the figures which he defines. Definitions may be conſidered as of two kinds; firſt, ſuch as ſerve only to explain the meaning of a word; but theſe

are not properly geometrical definitions, for from this no consequence can follow unless there be a mystical virtue in the name. Secondly, those from which consequences or properties do follow, which may be called geometrical to distinguish them from the others. The definition of a circle, given by Euclid, is of this kind for all the properties of a circle follow as consequences from it. And here the reader is to be admonished, that, upon this and such like occasions, his common notion of a circle is to be laid aside, and nothing admitted as a property of this figure, unless it can be shewn to follow from this definition.

Though Euclid in the arrangement of his principles has placed the common notions after the definitions, yet they are prior to them in the order of conception; and indeed if this is not attended to, some of his definitions will be unintelligible; for instance his definition of a straight line. He says a straight line is that which lies evenly to the points in itself. Now if I am to conceive nothing previous to this, respecting a straight line; what can I understand by this definition; or what can I infer from it? The reader will be just as much at a loss to conceive the meaning of the word *evenly*, as of the straight line itself. But if we consider this definition as an improvement upon the common notion of a straight line, (see com. not. 12.) every thing is very intelligible: for after a proper examination of this principle, that two straight cannot inclose a space; every body will infer, though not scientifically nevertheless very confidentally, that every straight line must lie evenly to all the points in itself; otherwise he certainly might have hopes at least of making two of them inclose a space. I would be rightly understood upon this point; nobody can imagine that it is my opinion, that Euclid intended that the one of these should be inferred from the other scientifically; but only that the definition expresses the conception, derived from two lines, reduced into a more simple form; though indeed he himself reasons from the common notion as will appear in the fourth proposition.

Dr. Hooke one of the greatest improvers of practical mathematics, was notwithstanding so sensible of the necessity of laying an accurate foundation in theory that he thus speaks concerning a point. *A point is that which has no part.* " This which some

" would

"would deem the most inconsiderable thing in the world, seems "yet the most difficult to be understood; no sense, or imagination "or fant'cy can reach it, nor words describe it, but by a negative, "to tell you what it is not: For it is not to be taken in the sense, "that the whole Earth is called a point in respect of the Universe, "nor in the sense that the end of a tapering thing is called a "point, as of a pin or needle; though they seem to be the smallest "things we know; because these latter may be said to have as "many parts as the fore-mentioned; for since all quantity is divi-"sible *in infinitum*, the least quantity may be divided as often as "the greatest, and therefore whatsoever is divisible must have "parts, and therefore none of these can be properly called a point, "in the sense here named, unless this point be understood to be "the *apex* of a mathematical cone or pyramid, where the super-"ficies of it is determined, for that will be a mathematical point: "But it cannot be supposed of a physical point, or material cone "or pyramid, for that will have extension and bluntness. And we "find that *microscopes* will make those points divisibe even to sense, "nay even almost to discover a new world in a point, nay there is "one now that affirms he has seen more than ten thousand living "creatures in the bigness of a very small sand, which itself indeed "is but a visible point to the naked eye; and each of those ten "thousand may have worlds within them. We know not the "limits of quantity, matter, and body as to its divisibility or ex-"tension; no imagination can comprehend the *maximum* or the "*minimum naturæ*, our faculties are finite and limited, and we "must content ourselves within the orb and sphere of their acti-"vity. And acquiesce in a negative definition, and understand if "we can somewhat that is smaller than the smallest, though that "be also improper; for in that which is not quantity, there is "neither smaller nor bigger, we must endeavour to understand "some what infinitely little, less than which there cannot be, "somewhat that has no bigness or extension or quantity, but only "position and respect to quantities circumjacent: From which, to "this or that body, there is a determinate length and distance; "and upon this account, whenever we endeavour to understand "this notion, our imagination will represent to us the smallest

 "visible

"visible body, as an exceeding fine sand, or a mite, or the point "of a needle, or the smallest visible body we have ever seen on "paper, or the like; which we must be content with, since the "fant'cy forms nothing but what is first in the sense, though it be "none of these. And in truth it can have no true definition that "will reach its essence. Analogous to this *point*, sign or nothing "in quantity is the *nought*, cifer or zero in number: The *never* "in time: The rest or quiet in motion. For as no aggregate of "points will ever produce a line or quantity; so the multiplication "of *noughts* or *cifers* will never produce a number; and as the "addition of *nevers* cannot make time, so the aggregate of *rests* "cannot produce motion. So that all these may not improperly "be called the *terminus* or bound, from which they all begin; so "quantity may be said to begin from a point or nothing: number "may be said to begin from *nought*, cifer or zero: time may be "said to begin from *never*: and motion to begin from rest. And "as the *minimum naturæ* may be said to be the first quantity; if at "least there be a *minimum* in nature, so an unite may express it in "numbers; *instant* in time, and moment in velocity. It may pos- "sibly be thought I have said too much of *nothing*, but yet it "seems to be of the greatest consideration in nature; for it seems "to be the beginning of every creature; even the greatest creatures "having been traced to begin from an atom or point, no eye or "sense can reach it; nor any understanding limit it; that the be- "ginning of a very large animal hath been seen alive, ten thousand "times smaller than a mite, may be proved, and yet how much "smaller it may have been is not determined."

It has been said of the musick of the spheres that it is so loud we cannot hear it: and of a mathematical point, according to this explanation, it may be said, that it is so simple it is impossible to comprehend it. This no doubt, must prepossess the reader with strange apprehensions of difficulty in the prosecution of a science which sets out so unaccountably; requiring us to admit as a principle what we cannot comprehend.

I would not have thrown this metaphysical bugbear in the way of the reader; only I was afraid lest he might stumble upon it of himself, with some hazard to his understanding, if he should pursue the

the metaphyfical analyfis of it, until he had no doubts left but that it was really incomprehenfible; efpecially if he happened at the fame time to difcover, that the fame ingenious fubtilties would alfo apply to lines and furfaces, by which means they might likewife be entitled to a place among the incomprehenfibles.

I fhall therefore beg leave to call his attention to a method of explanation very different from this, but which arifes very naturally from what I mentioned before viz. that the definitions fhould be confidered as fubfequent to the common notions; and introduced merely as auxiliaries to them. And in the prefent inftance, we have a very clear notion of three dimenfions in magnitude; though we always find them fo connected that it is impoffible to feparate them from one another. Let us therefore try to contemplate them together, but one after another. Every part of fpace that we confider has a fhape; for inftance the fhape of a room: this fpace the mathematicians call a folid; that which limits it or gives it this fhape is called a furface; which muft be confidered as having only length and breadth; becaufe if it had thicknefs alfo; then it would not be the boundary of the folid, but a part of the folid itfelf. Therefore the proper definition of a furface will be, that which hath length and breadth. But farther this furface has a fhape or is limited; that which limits it cannot have length and breadth; for then it would not be the limit of the furface but a part of it. Therefore the proper definition of a line is length without breadth. Laftly this line has its limits; which limit cannot have length, for then it would not be the limit of the line but a part of it. Therefore the proper definition of a point would be, that which belongs to magnitude, and has no parts, that is none of the common dimenfions, length breadth or thicknefs. Thus it is eafy to conceive how thefe definitions of a point line and furface are derived from the common notions of magnitude.

CHAP. VI.

The fame fubject continued.

AS the definitions which we have been explaining are never quoted or appealed to as tefts by Euclid, the learner might be ready

ready to conclude that they were placed here for no other purpose but to impress one with an idea of the difficulty of this science.

Though it be true that they are never used directly, yet by being placed where they are, they stop the mouths of impertinent critics, who would be ready to start objections which are immediately removed by these definitions.

In this chapter I intend to explain what is meant by a PLANE, a plane rectilineal angle, a right angle and an acute and obtuse angle; first considering what opinion every one might form of these, by examining those instruments or circumstances which would fix his attention to this subject.

To understand rightly Euclid's definition of a plane surface, it will be necessary to have recourse to the former supposition, that the definitions are improvements upon our common notions: for instance, let us suppose one at work, in the manner of a carpenter, to satisfy himself that a surface was even, suppose the surface of a table: he would apply a ruler in all directions from side to side; and finding that it touched the table in every point and in every direction, he would conclude that the surface was even. And now we shall find that the definition is nothing else, but giving the notion acquired by this operation, a regular form and sufficient extent,

For Euclid's plane surface which is commonly called a PLANE is this surface of the table, endued with a scientifical evenness instead of a mechanical one; and instead of being confined to any shape; of an unlimited extent. It is *in* or *upon* such a surface as this, that Eulid supposes all his lines and figures to be described and drawn, in his first six books: and this is the more carefully to be observed, because, though it is not mentioned in the demonstrations, yet there are many properties of lines and figures, which are not true, but upon the supposition that the lines are in the same plane, as I shall have occasion to observe.

And the plane rectilineal angle is the inclination of two straight lines to one another *in* or *upon* such an even surface as this, which meet together but are not in the same direction. They might be inclined to one another so that when produced they would intersect

or

or crofs one another; but until they have met they are never confidered as containing an angle.

This plane rectilineal angle may be of three different kinds; according to the different inclinations which two ftraight lines may have to one another. Take two pens, and run a pin through the extremity of the one, and at the fame time, through the other, at fome diftance from its extremity, fo that the one of the pens may make two angles with the other, on each fide one. Turn the pens into the fame direction, and making one of them revolve, and then obferve its progrefs in revolving until it takes the fame direction on the other fide: and during this whole revolution there is but one fixed or determinate pofition of the lines with refpect to one another; and that is when the revolving line has the fame inclination to the other on each fide; and thefe equal inclinations are called right angles; and the other two have their names according as the angle is greater or lefs than this.

The reader has been very inattentive to my directions upon this head, if he does not perceive, that there is an unlimited variety both of acute and obtufe angles, but that there is but one right angle.

And hence by the bye it is obvious that Euclid formed his definitions from common notions, as he reckons this one; that all right angles are equal, probably as a hint to the reader, that he might difcover from what fource the definitions were derived; becaufe this common notion may be regularly demonftrated from the definition of a right angle, and therefore muft have been placed where it is for fome indirect purpofe. Upon the whole then we may conclude, that Euclid's definitions are not the imaginary phantoms which the metaphyficians have reprefented them: but notions derived from material objects, and new modeled, not for the purpofe of empty fpeculations, but, that they might be applied to the fame kind of objects which firft fuggefted them, upon a more accurate and enlarged plan.

CHAP.

CHAP. VII.

Of postulates.

THIS ſcience has now acquired ſuch an extent and accuracy, by improving the common notions into definitions; that our former inſtruments are now become inadequate to our purpoſes and views. For our notions both of a ſtraight line and circle are ſo refined and enlarged; that no inſtrument is ſufficiently accurate or extenſive for performing theſe two problems, the drawing a ſtraight line and the deſcription of a circle.

Here the caſe of the ſcience appears deſperate; and the author reduced to the neceſſity of giving up his high pretenſions to accuracy and univerſality, by ſuffering the ſcience to fall back again into its original ſtate, as far as reſpects its inſtruments at leaſt. And I am certain the artifice uſed upon this occaſion to preſerve the dignity of the ſcience is by no means generally underſtood; and this will be confirmed beyond contradiction by attending to the fooliſh and childiſh reaſons generally given for admitting theſe poſtulates, alledging that it is ſo eaſy to conceive how they may be done, that they are to be admitted without the leaſt ſcruple or heſitation. That a thing may be eaſily done is ſurely a very bad reaſon for neglecting to give rules for doing it: and if that had been the caſe in the preſent inſtance, I am pretty confident theſe commentators would have been ſaved the trouble of their apology. But the opinion of Newton I ſuppoſe will be deciſive upon this occaſion, and we find him expreſſing himſelf directly to our purpoſe as follows. "Nam et linearum rectarum et circulorum deſcriptiones, in quibus *geometria* fundatur, ad *mechanicam* pertinent. Has lineas deſcribere *geometria* non docet ſed poſtulat. Poſtulat enim ut tyro eaſdem accurate deſcribere prius didiceret, quam limen attingat *geometriæ*; dein, quomodo per has operationes problemata ſolvantur, docet; rectas et circulos deſcribere problemata ſunt, ſed non geometrica. Ex *mechanica* poſtulatur horum ſolutio, in *geometria* docetur ſolutorum uſus. Ac gloriatur *geometria* quod, tam paucis principiis aliunde petitis, tam multa præſtet.

After ſuch an authority the judicious reader will be ready to agree with me, that theſe problems are taken for granted, not becauſe

caufe it is eafy to conceive how they may be done, but for a much better reafon, becaufe it is impoffible to do them by any method confiftent with the principles of this fcience.

But even in this, which muft have been a matter of no fmall difficulty, our author has done every thing in the power of man, to render this unpromiffing part as correct as poffible, by making it put on a very refpectable figure indeed, giving it fuch an air, as difcovers but little of its mechanic original,

An ordinary genius would probably have given directions for drawing a ftraight line and defcribing a circle, with a ruler and compaffes; and when the limited nature of his inftruments was mentioned as an objection to his fcience, and that he could not with propriety extend his conclufions beyond a fheet of paper; he would then probably propofe a chain and poles by the affiftance of which he might undertake to enlarge the fcale of his operations; yet with waggon loads of fuch inftruments he never could hope to produce one fcientific conftruction.

But Euclid has acted with more judgement, and never would be acceffory to the opening a door for admitting upon his fcientific ftage the whole tribe of mechanics, with the feveral implements of their trade. He throws them two problems to perform, but this is to be done behind the fcenes, without exhibiting their mechanical inftruments, declaring that upon every future emergency, he is determined to ftand or fall by his own principles.

CHAP. VIII.

Of the inftruments made ufe of for communicating geometrical knowledge.

ALTHOUGH language be a general medium for communicating knowledge of every kind; yet particular fubjects require the introduction of fome auxiliary inftruments. Unlefs the nature of the thing is fuch, that it cannot fail of itfelf to make a ftrong impreffion upon the mind, or that it is of no great confequence whether it be particularly examined or not, it will be found difficult to command the attention; and particularly, which is abfolutely neceffary in a chain of reafoning, to make the thoughts of

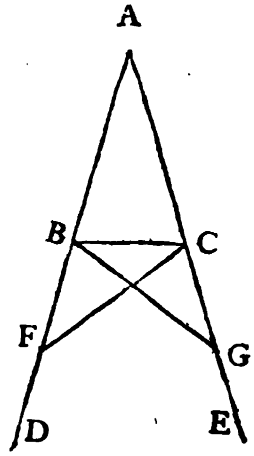
 the

the reader and writer follow each other in the same order, with a certainty that they are both thinking of the same thing at the same time. The art of logic is an instrument of this kind, but the great diversity of opinions prevailing among mankind is a sufficient proof that this instrument is either neglected or misapplied. Indeed that wonderful uniformity of opinion which has been universally found among all nations, and in all the different ages of the world upon geometrical subjects, cannot fail to draw our attention to this singular circumstance; and make the learner desirous to be let into the secret of this harmony, as far as one ignorant of the subject can be supposed capable of entering. And this will be found to proceed from the method of demonstration to which geometricans have tied themselves down; and their being possessed of an instrument, perfectly adequate to the purpose of communicating their knowledge to others, and by which they are able infallibly to examine their own. And this instrument in geometry is figures or diagrams: which Euclid expresses by the different names of σχῆμα, καταγραφη, εἶδος. The first is his general term, the second he commonly applies to a figure which has been described or at least compleated; and the third commonly to similar figures.

Now these diagrams or figures may be made so exactly to resemble the subject matter of any proposition, that, if we think at all, it is impossible to mistake the order of thinking, which the author has prescribed, or to draw a different conclusion from that, to which he intends to lead us. I say a different; because the figure itself may lead us to a limited or partial conclusion; the remedy for which I shall explain at length in the next dissertation.

The simplest kind of rectilineal figure is the triangle; the different parts of which the learner ought to be familiarly acquainted with; and to keep in his mind, that besides the triangular space itself; there are six different magnitudes which go to the making it up; viz. three sides and three angles; each of which ought to be particularly attended to. It will be useful also to observe; that the triangles ABG, ACF though they have several parts in common, are nevertheless to be considered as much as distinct and different triangles, as if separated from each other by the distance of a thousand miles.

The



The three ſides of the triangle ABG are AG, AB, BG and the three angles are ABG, AGB, GAB: and of the triangle ACF the three ſides are AF, AC, CF; and the three angles ACF, AFC, CAF and the angle at A or DAE is ſaid to be common to the two triangles; and therefore they have one angle equal to one angle; and it is ſaid to be contained by the ſides FA, AC when it is conſidered as an angle of the triangle ACF: but when it is conſidered as an angle of the triangle ABG it is ſaid to be contained by GA, AB.

Again FBC and BCG are alſo two diſtinct triangles: and the three ſides of the triangle FBC are BF, FC, CB and the three angles BFC, FCB, CBF: and the three ſides of the triangle BCG are CG, GB, BC; and the three angles CGB, GBC, BCG: theſe triangles have a common ſide viz. the ſide BC; and this ſide is ſaid to be extended under the angles BFC and BGC. Theſe things ought to be well underſtood and ſtrongly impreſſed upon the memory.

But farther, we find that the two triangles ACF and BFC have a ſide and an angle common to both, viz. the ſide FC and the angle AFC which is the angle BFC; alſo the two triangles ABG, BCG have a ſide and an angle common to both, viz. the ſide BG and the angle AGB which is the angle CGB.

I have now explained at ſome length the original and nature of the geometrical principles; and the inſtruments made uſe of for communicating this kind of knowledge, taking notice at the ſame time of ſome of the moſt unuſual forms of expreſſion and which are apt to perplex the learner at his firſt ſetting out.

DISSERTATION II.

Concerning the nature and extent of geometrical demonſtration.

IN the former diſſertation, as much care as poſſible has been taken to avoid the ſuggeſting any circumſtance to the reader's imagination, which will not be neceſſary in the very firſt ſteps to be taken in this ſcience. I muſt therefore beg he will not conſider what has been ſaid as words of courſe, merely to ſcrape acquaintance, or any attempt to diſplay my own learning, but as ſeriouſly intended for his improvement. Nor would I have it thought that this proceeds from any want of inclination to be in the good graces of my reader, much leſs from want of capacity; for with very conſiderable deductions both of thought and trouble, like ſome other commentators I could have flouriſhed away, in an explanation of the eleventh common notion, upon the properties of the aſymptotes of the Hyperbola &c; if my intention had been to raiſe the admiration of the reader, rather than to fix his attention. Having therefore reſolved to ſacrifice every ornament to this ſingle conſideration; the learner, it is to be hoped, will endeavour to convince himſelf, or rather ſome more impartial judge, that he underſtands every thing mentioned in the laſt diſſertation before he proceeds to this; in which he will find the nature and end of geometrical demonſtration laid open with all the ſimplicity and perſpicuity which the commentator could give it; and with as much particularity as the patience of the reader, a faculty rather apt to be diſcompoſed by much exerciſe, could well be ſuppoſed to bear.

CHAP.

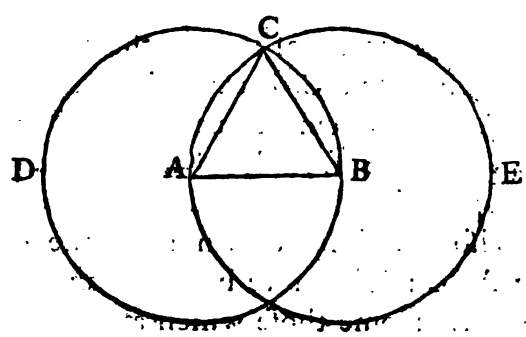
CHAP. I.

Containing an explanation of the two first propositions.

THE two first propositions are thus explained by Dr. Hooke. "Euclid having premised his principles, he begins his method "of demonstration, in which he takes no more for granted than "what he hath already laid down as easy and self evident. His first "proposition then is upon a right line given to make an equilateral "triangle. He hath defined in the fourth definition what he "means by a right line, namely that which lieth straight between "the two extremes of it which are points; and what he means "by an equilateral triangle, namely, such a one which hath all its "three sides equal to one another."

"This first proposition is a problem, which explains a way how "to do and perform the thing required, as well as shews how to "manifest the truth and certainty of the thing done: It contains "therefore and shews a double invention, without which or some "such other thing, the proposition can neither be done nor demon-"strated; which inventions are called mediums or means by which "we attain to the end propounded or desired. The end here "sought is how from the ends of a line given to draw two other "lines, each equal to the given line, which shall meet in one and "the same point. It is certain that these lines must begin from "the ends of the first line given; but which of these to draw first, "and which way, and with what inclination to the former line, "that is with what angle, that is not yet known, and some inven-"tion must be thought of, how to direct our ruler to draw it. "Well how shall this be done, since there may be infinite of "lines drawn from each of those points, which shall every one of "them be equal to this line given? How then shall we among "those infinite or indefinite number chuse out the right? 'tis im-"possible without some invention. Our author therefore helps "you to one, and one which you have already granted to be feasi-"ble in the third petition. Upon the center A, and distance AB, "draw a circle, says he, BCD; what then? To what purpose; Why

“ Why this circle then will give you a line, in which are contained
“ all the points or ends of the infinite lines, which may be drawn
“ from the point A any ways that shall be equal to AB. How so?
“ Why by the fifteenth definition you are taught, that a circle is a
“ plain figure bounded by or contained within one curve line,
“ which is called the circumference; to which every right line
“ drawn from a point in the middle, which is called the center,
“ are equal to one another. But what are we yet the wiser? How
“ do we know which of these infinite lines we are to draw? To
“ which of these infinite points that are in this circumstance? To
“ know this, you must do the same thing upon the point B: that
“ is, upon the point B and distance BA, draw or describe the circle
“ ACE, which will give you all the possible infinite points in that
/ne “ plain; to the which from the point B right lines may be drawn
“ equal to BA. Now then since these circles contain all the pos-
“ sible points of the lines equal to AB or BA that can be drawn
“ from A or B. It follows that where those circles intersect, there
“ only must be the point to which those lines may be drawn;
“ namely at C and at the other point of intersection and no where
“ else soever. Drawing therefore lines from A and from B to either
“ of those points as to C as AC, BC; you have done the thing
“ that was propounded; namely upon the line AB given you have
“ made an equilateral triangle ABC; which was desired. This is
“ the first part of the pro-
“ blem, and indeed the
it difficult + “ ~~difficultest~~ to find out;
“ namely how to do the
“ thing required; and in
“ this part lie the greatest
“ difficulties of mathema-
“ tical knowledge, to wit,
“ in finding out thepro-
“ per and true mediums



“ or means to perform the problems required to be done, which
“ for the most part are of the same nature with this, and con-
“ sist in the finding out the position of a point; for this pro-
“ blem might have been thus worded. A right line being given

“ as

" as AB, to find a point as C, to which lines being drawn from
" the points A and B, they ſhall each of them be equal to one
" another, and to the line AB which is given: or two points A
" and B being given to find a third point as C, which ſhall have
" the ſame diſtance from A and from B that they have from one
" another. But our author not having given any definition of diſ-
" tance or equality, otherwiſe than may be collected from equality
" of the ſides of ſome figure; or of the rays or lines drawn from
" the center to the circumference of a circle, he chuſes rather to
" make uſe of an equilateral triangle to find out that propriety of
" a point ſo poſited."

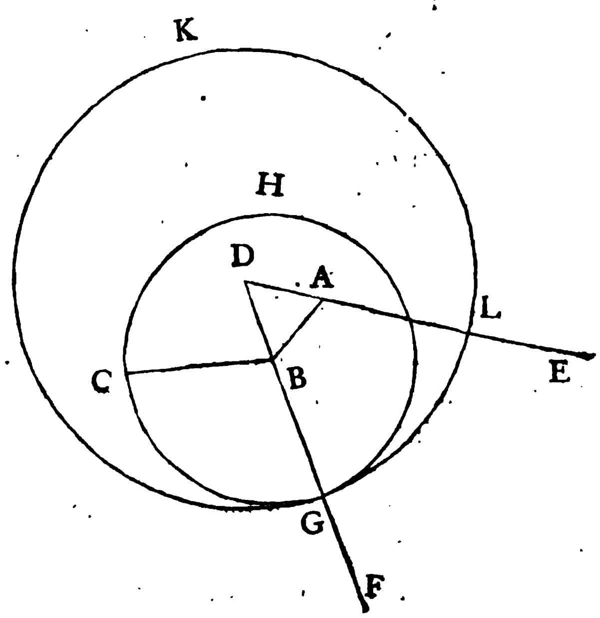
" The ſecond part then of the propoſition is the demonſtrative
" part thereof, namely, to prove from the principles already laid
" down and granted and aſſented unto for true and certain, by a
" clear chain of reaſoning and deduction, that theſe lines AC and
" BC are each of them equal to AB and ſo equal to one another;
" and conſequently that the figure ABC bounded and limited by
" them is an equilateral triangle, according to the deſcription of
" that figure in the twenty fourth definition. The next thing then
" to be invented or found out is the medium or means of demon-
" ſtrating it to be ſuch; for this we have two. Firſt, the definition
" of the properties of the lines from the center of a circle to the
" circumference in the fifteenth definition, that they are all equal
" to one another. And ſecondly we have the firſt axiome, thoſe
" which are equal to one other are equal to one another. Firſt AB
" is equal to AC, becauſe they are right lines drawn from the center
" A to the circumference BCD; for by the fifteenth definition, as
" I ſaid, all ſuch lines muſt be equal. Next BA and BC muſt
" upon the ſame account be equal to one another, becauſe they
" alſo are lines drawn from the center B to the circumference
" ACE: Therefore both AC and BC are equal to AB; but by the
" firſt axiome thoſe which are equal to one other are equal to one
" another, therefore AC and BC are alſo equal to one another.
" Therefore the three ſides of the figure ABC; namely, AB, AC,
" and BC are equal to one another, and conſequently bound an
" equilateral triangle according to the twenty fourth definition,

" therefore upon the line AB given, an equilateral triangle is made, " which was the thing to be done and proved."

" THE second proposition is also a problem.

" From a point given to draw a right line, which shall be equal " to another right line given. For the performing of this propo- " sition, it being a problem, there is need of two mediums to be " invented or found out; the first is for the doing of the thing re- " quired; and the second for the demonstration of it; and both " these are to be fetcht out of the principles already agreed to, or " from the truth evidenced in the preceding proposition: For we " are not to suppose any thing further known in this science, and " therefore much less are we to make use of it. Searching there- " fore our store, we have no other medium to make a line equal to " a line, than first by the help of a circle, defined definition 15. " which by the third *postulatum* is granted to be describable upon " any center and at any distance: Or secondly, by an equilateral " triangle described by definition 24, which we learnt how to make " by the preceding problem: For as to the equal sides of an isos- " cles triangle, definition 25, or the equal sides of a square, a pa- " rallelogram or oblong square, Rhombus or Rhomboeides descri- " bed definition the 30, 31, 32, and 33. Though their properties " are there defined, yet we are not taught how to make them as " yet, and consequently can make no use of them, as *media* to " perform the thing required to be done by this problem. Nor " are we to suppose, that the length of the line given may be taken " by the help of a measure or a pair of compasses, and transferred " to the point given; because those are first not mentioned in the " principles laid down there, which you are to make use of, and " of no other, till they be accepted for principles undeniable: For " this is not yet granted, that you can with compasses, take a true " length of a line, much less that you can transfer it and set it in " another place. But you have granted that 'tis possible, upon a " point given at any distance, to describe a circle, or suppose it so " done, which is sufficient for the demonstration, that being the " principal thing aimed at by our author, namely, to lay open to " the understanding the resons and grounds of the properties of " quantities so and so qualified, that you may plainly see how and

“ for what cauſe things are thus or thus and cannot be otherwiſe.
.“ For as to the moſt practicable and expedite ways of doing thoſe
“ things mechanically, and for other uſes, *that* belongs to another
“ part of mathematics; namely to the practical part thereof, which
“ is called practical geometry, or mechanical geometry, which
“ ought not to be learned till this be firſt known; but this which
“ our author treats of is ſpeculative geometry, and principally aims
“ at demonſtrations or explaining the proprieties of quantities to
“ the underſtanding. You ſaw clearly by the former propoſition
“ why ABC was an equilateral triangle; and there could be but
“ two ſuch made upon the line AB in the ſame plain; there being
“ but two points where in thoſe circles cut each other; thoſe cir-
“ cles determining all the lines equal to AB that can be drawn
“ from the points A and B. His way then of performing this
“ problem is this: Let the right line given be BC and the point
“ given be A; from
“ which point a right
“ line is to be drawn
“ equal to the line BC.
“ Firſt draw a line from
“ B to A, which is
“ granted poſſible by
“ the firſt poſtulatum;
“ then by the former
“ propoſition upon this
“ line, BA make an equi-
“ lateral triangle BAD:
“ then upon the center
“ B and diſtance BC
“ deſcribe a circle, as
“ CGH by the third
“ poſtulatum; then by



“ the ſecond poſtulatum produce DB to F; then upon the center
“ D and diſtance DG draw the circle GKL, then as before, pro-
“ duce the line DA to E; there ſhall AL be the line required to
“ be drawn from the point A equal to the given line BC.”

“ This

"This is the conſtruction of the problem, or the preparing of "the propoſition fit for demonſtration, by which you may clearly "underſtand the reaſons of it, deduced from the few principles "already laid down: For firſt, that BC is equal to BG is clear "from the fifteenth definition, which determines the propriety of "equality of the rays of a circle. Next that DBG is equal to "DAL, is as clear from the ſame definition, they being both rays "or lines drawn from the center D to the circumference GKL by "the conſtruction premiſed. Thirdly that DB is equal to DA, is "clear from the conſtruction; for DBA is an equilateral triangle, "two of whoſe ſides DB and DA are. Now by the third axiom "or common notion, if from equal quantities you take equal quan-"tities, the remainders ſhall be equal; if from DL you take DA; "and from DG, DB; the remainder AL ſhall be equal to the re-"mainder BG; but BC is alſo by the conſtruction equal to BG; "therefore ſince by the firſt axiom theſe two quantities which are "equal to one other quantity are equal to one another; therefore "BC and AL, being equal to BG, are equal alſo to one another; "therefore from the point A the line AL is drawn equal to the "line BC; which was the thing to be done and proved."

"Now though this way of demonſtration and reaſoning may "ſeem tedious and too long to detain the mind and attention in "the finding out the proprieties of quantities, yet 'twas the way "made uſe of by the ancients. And 'tis altogether neceſſary, eſpe-"cially in the beginning of this ſtudy, to accuſtom the mind to "attention and circumſpection, that it may receive nothing for "truth but what it ſees clearly by the reaſons and cauſes of it, "that thereby the mind may acquire an habit of intention, and "of examining the whole chain of conſequents from the firſt prin-"ciples to the truth evidenced. For the want of which, ſome "ſmall error perhaps may ſlip into the mind under the appearance "of truth, and thereby make all the ſubſequent reaſonings and "deductions unſound; and 'tis very much harder to clear and free "the mind from it when once received, than to prevent the recep-"tion thereof. There cannot therefore (in this ſtudy eſpecially, "not now to mention any other, where it is poſſible it may be "altogether as convenient, nay neceſſary) there cannot I ſay

" therefore be as I conceive too much circumſpection and caution " uſed in admitting principles, and furniſhing the mind with the " true grounds of knowledge; becauſe for the moſt part we are too " prone to take up every thing we hear upon truſt, without exami- " nation: we are too apt to run away with a thing, and think we " know it and ſee it clearly before we are ſure we do; and are " impatient of delay in examining and conſidering; whereas if the " mind be a little at firſt accuſtomed to this leiſurely and ſtrict " way of reaſoning, after it has got a habit it will make as much " diſpatch in receiving things with ſufficient examination, as ano- " ther ſhall without it. And the patience only is needful for the " moſt part at firſt to beget attention; nor is it peculiar to this " acquiſition alone; but we ſee it neceſſary, and practiſed in many " other things where a good habit is to be acquired; as in reading, " writing, muſic, drawing, and moſt other manual operations: " The roots and beginnings of knowledge, and practice too, are " bitter and tedious, but the fruits are ſweet and pleaſant; and " whoſoever attains the end, will never repent the time ſpent in " the beginning."

CHAP. II.

The ſame ſubject continued.

IN the preceding chapter I have given Dr. Hooke's explanation of the two firſt propoſitions, in order to ſhew the reader that I am not ſingular in my opinion, that Euclid is not to be underſtood, unleſs the learner beſtows that attention upon the firſt principles, which may enable him to carry along with him their full meaning and import. It is true that ſome confuſed facts concerning the properties of figures may be picked up by a very careleſs peruſal of this author: but whoever is ſatisfied with this, had better look for his knowledge ſome where elſe; becauſe he will meet with a great many impertinent interuptions to his ſcheme from the ſeveral ſteps of the demonſtration; and rather content himſelf with a kind of law evidence, by reſolving to conſent to every propoſition which is delivered as truth by two or more credible authors.

Although

Although I approve in general of what Dr. Hooke has advanced, yet I am convinced that a more minute consideration of the different parts of each proposition will be necessary for the student who would wish to leave his prejudices behind him as he advances. I shall therefore point out these circumstances, which he might be apt either to overlook or mistake. Supposing the first proposition of Euclid carefully examined, he will find it taken for granted that the two circles cut one another, and that this supposes the circles to be described upon the same even surface, or according to the geometrical language in the same plane, otherwise there could be no such point as C. Again the supposing C to be a point implies that the two lines, the two circumferences of the circles have no breadth, otherwise the point would have parts. Likewise by supposing the triangle ABC to be an assignable or determinate magnitude implies that the points A, B, C have no magnitude for if they had parts various triangles differing both in sides and angles might be described, and it would be impossible ever to arrive at a determinate conclusion.

But farther, the reader is to consider that he has been reasoning all this while, upon a particular straight line of very inconsiderable length; He is therefore next to inquire, whether his constructions and conclusions are also particular. That they are not will appear from this, that no consequence is supposed to follow from the straight lines having any particular length, nor is any construction undertaken upon that supposition; only that the line be finite that is, that we know its two ends.

But it happens rather unluckily, that though the scientific construction be general we are nevertheless forced to take up with a particular one; which is very apt to create prejudices unless our attention be every now and then called to the general construction: we must remember that, though we work with a ruler and compasses, the science knows no such instruments. Let us suppose the line AB ten miles in length; and that the same construction is to be performed; we shall now get beyond the objects of our senses; but if we have bestowed the necessary attention to the problem, the understanding will have as clear and distinct a perception as it had before, when the line was perhaps not above three inches long.

Our

Our author's plan obliges him ſtrictly to prohibit the uſe of any particular inſtruments; ſtraight lines and circles are ſuppoſed to be drawn and deſcribed by mechanical operations; and we are left to gueſs in what manner. It will however be prudent to acquire ſome one ready mechanic way at leaſt of conſtructing every problem. For I always ſuſpect any pretenſions to general truths which have not been collected from an examination of particulars. The ruler and compaſſes are ſuch exact minature repreſentations of the two poſtulates, that they are to be preferred to every other inſtrument for aſſiſting the imagination to follow Euclid's conſtructions: only the compaſſes are to be confined to their proper uſe, always to deſcribe circles, but never to meaſure diſtances. Indeed it is not always neceſſary, and often would introduce confuſion if the circles were compleatly deſcribed: it is therefore ſufficient to deſcribe as much of them as is neceſſary; as for inſtance, in the future application of this propoſition, it is obvious that it will be ſufficient to find the point C; becauſe the point where the circumferences of the circles cut one another gives us every thing neceſſary for the conſtruction.

We come now to the ſecond problem which is of a more complicated nature than the firſt; and yet there are very few, upon reading it, who do not fall in with the notion that Euclid has taken a great deal of unneceſſary pains to do what a ſingle opening of the compaſſes would have performed with equal or perhaps more exactneſs. But conſidering the ſcience *only* with a view to practice, the affairs of mankind are carried on upon a much larger ſcale than to be managed with a ruler and compaſſes; and we have only to ſuppoſe the line BC a mile in length, and this will anſwer the objection ſufficiently. The underſtanding is capable of reaching the general concluſion delivered in this propoſition; why then ſhould it be fettered with inſtruments? Euclid's contrivance is admirable and ſuited to the dignity of the human mind. According to his plan we are guided by inſtruments, but neither confined nor loaded with them.

After the ſtudent has fixed in his memory the conſtruction and demonſtration of this ſecond propoſition, it is more than probable that his concluſions will be limited to the particular figure which he

be made use of and to that very position which the lines happened accidentally to take.

This is a very general prejudice, and will be removed by attending to the following directions. Let the learner consider whether DA and DB produced will cut the circle CGH: and he will find that DB must cut it as it passes through its center, but that DA's cutting it depends upon the length of AB. Change the position of the point A and repeat the construction and thus you will learn what lines have a fixt position or only an accidental one: describe the equilateral triangle ABD upon the other side of the line AB and compleat the figure; also join the points A and C instead of the points A and B and repeat the same constructions; and by proceeding thus you will be able to conquer the prejudice of sense and to acquire something like a scientific view of the problem. For you will find no consequence deduced from BC's having any particular length or the point A any particular position.

But the third proposition will afford an opportunity of taking a somewhat different view of these two propositions. It is a problem. Two unequal straight lines being given to cut of a part from the greater equal to the less. The simplest case of this proposition would be, when both the lines are drawn from the same point: for making that point the center, and the shorter line the radius, the circle so described would solve the problem: but if the lines had any other position, a very different apparatus would be necessary; we must learn how to put a line at the extremity of the longer equal to the shorter line; and this cannot be done before we have learned how to describe an equilateral triangle. I would therefore propose it as an exercise for the student, and his success would be a proof that he was master of the subject, to set out upon a supposition that the book began with this third proposition, which will now include the first and second; and by this process he will have their connexion with each other strongly impressed upon his mind. His figure will consist of five circles, and their use is to determine four points: such a point as C in the first proposition, G and L in the second; and E in the third.

Whoever attempts to communicate knowledge to mankind, must write upon the supposition of a certain degree of improvement

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in thoſe to whom he addreſſes himſelf, otherwiſe his book can have no determinate end to anſwer. Euclid ſuppoſes his reader above the prejudice of ſenſe, and to have the ready uſe of his underſtanding, with a due command of atttention; and upon this ſuppoſition he has ſaid every thing neceſſary to convey the fulleſt information to his reader. But I write to a different ſort of people, to ſuch as are immerſed in the prejudices of ſenſe, and at the ſame time very thoughtleſs; to thoſe whoſe underſtandings are not diſpoſed to attend to the call of reaſon without frequent admonitions; and this is my apology for begging the ſtudent's attention to one thing more, before I finiſh this chapter; he will always find ſomething ſuppoſed or given in every propoſition, and perhaps nothing will contribute ſo much to a right underſtanding of the propoſition, as a diligent enquiry into the uſe made of the *data* or ſuppoſition, as for inſtance whether the reaſoning in the third propoſition, does not depend upon the ſuppoſition that AB is longer than C; and where the reaſoning would fail if that were not the caſe. Our author was very confident that his reader would know, that, without this part of the ſuppoſition, there never could have been ſuch a point as E: but I am affraid that mine truſted to his ſenſes for the real exiſtence of this very point; dont I ſee, ſays he, that the circle cuts it.

I ſhall conclude this chapter by reminding the reader again, that the uſe of the two circles in the firſt propoſition is to find the point C: and in the ſecond, that the equilateral triangle is deſcribed to fix the point D, which is to be the center of a future circle; whoſe radius DG is determined by the deſcription of the circle CGH; and laſtly that this circle itſelf is deſcribed to find the point L. And thus I leave the ſtudent to purſue his own meditations upon theſe three propoſitions, only advertiſing him, that, if he thought them eaſy upon the firſt reading, and ſtill perſiſts in the opinion that he then underſtood them, I am certain he knows nothing of the matter.

CHAP.

CHAP. III.

Concerning hypotheſes.

THE fourth propoſition is of a different kind from the three firſt; and is called a theorem. In the problems ſomething is required to be done: a problem therefore may be divided into two parts the conſtruction and demonſtration: for after ſhewing in what manner it may be done; it is neceſſary to prove that this is agreeable to thoſe principles which the nature of your ſcience obliges you to adhere to. Thus in the third propoſition it is totally inconſiſtent with the principles of this ſcience to take the line C in a pair of compaſſes; and by this means cut off AE from AB; for whatever inſtruments we uſe, it can be allowed to do nothing more by their aſſiſtance than to draw a ſtraight line and deſcribe a circle; the compaſſes then may be uſed in deſcribing a circle; but it can do nothing elſe without encreaſing the number of our mechanical problems, which Euclid has avoided with the greateſt care; and here it is to no purpoſe to alledge that it can perform the one as accurately as the other: but I deſire it may be obſerved that it is not admitted upon the ſcore of its accuracy, but as keeping in ſome reſpect to the idea of the poſtulate; for its merit in point of accuracy is realy nothing.

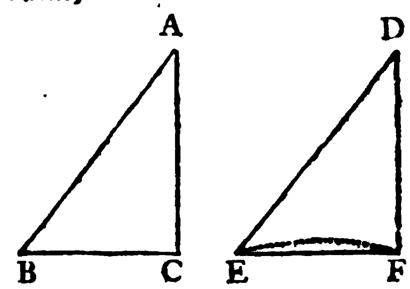
If geometrical knowledge could be communicated in the form of problems it would make a readier and ſtronger impreſſion upon the mind of the learner: for in the conſtruction of a problem the ſtudent is not barely a ſpectator but has an active part aſſigned him, to keep up his attention: but when a theorem is preſented to his conſideration, unleſs he has learned the uſe and management of an hypotheſis, he will be but an indolent and inattentive ſpectator; and to tell ſuch an one that the conſequence alluded to follows from the ſuppoſition is to addreſs him in a language which he does not underſtand. But for a more particular explanation I ſhall analyſe the fourth propoſition, which runs thus;

If two triangles have the two ſides equal to the two ſides, each to each; and have the angle equal to the angle, the *angle*, contai-

ned by the equal ſtraight lines: They will alſo have the baſe equal to the baſe; and the triangle will be equal to the triangle; and the remaining angles will be equal to the remaining angles, under which the equal ſides are extended, each to each.

This general enunciation of the propoſition will convey but a very indiſtinct perception of its meaning to a learner, eſpecially upon a firſt reading; this however may be improved and rendered diſtinct by the aſſiſtance of a figure, repreſenting two triangles in the ſuppoſed circumſtances: but here again a new difficulty ariſes, as this hardly ever fails to bring a prejudice along with it: for the triangles repreſenting not only the ſuppoſitions, but at the ſame time the inferences which are ſaid to follow from them; the ſtudent is at a loſs to diſtinguiſh what is given from that which is to be inferred from it; becauſe in all probability he will look upon the figure itſelf as the only ſource from which his knowledge is to be derived; and then he is as well convinced that the concluſions are true as the ſuppoſitions; and cannot conceive what it is he has to demonſtrate. And unleſs he be qualified to lay a proper ſtreſs upon the ſuppoſition, the demonſtration muſt appear to him an idle abuſe of words calculated only to perplex his underſtanding, eſpecially if he has already fixt his opinion by his compaſſes and other inſtruments. It is not eaſy to deviſe a ready remedy for this error; ſo that a man runs a riſque either of having no opinion concerning what is propoſed in the propoſition or an abſurd one; for of all the difficulties attending the acquiſition of the ſcience, this is the hardeſt to be got over. To reaſon accurately from a ſuppoſition is no eaſy matter; attention and habit will do a great deal, but above all a proper ſenſe of the difficulty of it. In the problems which we have already conſidered, the ſtudent has only to attend to the works of his own hands, in the firſt propoſition the ſtraight line AB is given him; but the two circles, and the two ſtraight lines AC and BC are his own manufacture; and likewiſe in the ſecond ſo is the whole figure except the point A, and the ſtraight line BC: this makes a diſtinction which nobody is ſo thoughtleſs as to overlook. But in a theorem like this fourth propoſition; where one has nothing to *do* but to *think*, the caſe is very different, eſpecially if the figure, which is intended to direct him,

him, by miſtaking its uſe, ſhould become the occaſion of ſeveral prejudices. But to be more particular,



Let ABC and DEF be two triangles; having the two ſides AB and AC equal to the two ſides DE and DF; each to each, viz. the ſide AB equal to DE; and the ſide AC equal to DF; and the angle contained by BAC equal to the angle contained by EDF.

This now is the hypotheſis or ſuppoſition, which is to be conſidered as abſolutely certain; and a principle from which we are to reaſon as confidently, as from the firſt common notion that magnitudes, equal to the ſame, are equal to one another. Whatever might be collected from looking at the triangles, or by any other means, is to be entirely neglected; and not merely neglected, but even ſhunned as leading to the moſt pernicious errors. The ſtudent has only to examine the hypotheſis, one part after another, making uſe of the figure only to aſſiſt him in comprehending its meaning, which he might ſet about in ſome ſuch manner as this, repeating to himſelf, the two ſides I find are equal, but not any how; for they are ſaid to be equal each to each; the ſides of all triangles are inclined to one another; but here a particular inclination is ſpecified; they are ſaid to be equally inclined: this the attentive reader will comprehend perfectly, and be able to ſay I underſtand now what the author means; he affirms that all triangles which agree with one another in the circumſtances above mentioned cannot poſſibly fail in giving us the ſame conſequences. But let us ſee what conſequences he ſays will follow from his ſuppoſitions— They will alſo have *the third ſide or* baſe BC equal to the baſe EF; and the whole triangle ABC equal to the whole triangle DEF: and the remaining angles —*ſtop*, why that epithet remaining? Why, becauſe only one angle in each triangle being equal by the ſuppoſition, are there not then two in each triangle which remain to be conſidered? But to proceed, the remaining angles ſhall be equal to the remaining angles not both taken together, but each

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to each. EACH TO EACH, this though ſomewhat particular nevertheleſs admits of a latitude ſufficient to perplex an ordinary underſtanding; and without ſome additional circumſtance, muſt remain unintelligible even to the acuteſt. But the angles are particularly deſcribed and alſo fixed; becauſe they are to have the lines which are equal by the ſuppoſition extended under them; that is the angles under which AB and DE are extended are to be equal, as alſo thoſe under which AC and DF are extended; viz. the angle ACB or BCA equal to the angle DFE or EFD; and the angle ABC or CBA equal to DEF or FED. Now theſe are ſaid to be the conſequences which will certainly follow from the forementioned ſuppoſitions.

But it will be too much for the learner to attempt the whole demonſtration at once; let us therefore ſee what conſequence will follow from each part of the ſuppoſition taken ſingly. The principle by which they are to be examined is the eighth common notion, magnitudes which apply themſelves exactly to one another are equal. It is here ſaid that AB and DE are equal. What do you underſtand by this word *equal*? Certainly it muſt mean, that, if the ſtraight lines AB and DE be properly placed, they will apply themſelves exactly to one another. But what is the proper way of placing them? Put the point A upon the point D and the line AB upon DE; obſerve all this may be done whether the lines be equal or unequal; but then the point B will apply itſelf to the point E only when the lines are equal; for when AB is longer than DE the point B will be found beyond E; and when AB is ſhorter than DE the point B will be found between D and E. After a careful examination of theſe two particular lines; let us ſuppoſe each of them a thouſand miles in length; here our ſenſes forſake us, but ſurely our underſtanding unleſs it be weak indeed, will give us as poſitive a concluſion as in the other caſe; ſo that we may conclude univerſally that whatever be the length of AB provided only AB and DE be equal; and the point A upon the point D, and the line AB upon DE; the point B will always apply itſelf to the point E; which was the firſt conſequence to be examined. Alſo if the point A be upon D and the line AC upon DF; the point C will apply itſelf to the point F, as is obvious

for

for the ſame reaſons; and that will always be the caſe whatever be the length of AC.

We come next to conſider in what circumſtances it is poſſible to place AB upon DE; and at the ſame time AC upon DF: Now this can always be done, if the angle BAC be equal to the angle EDF; becauſe when the point A is upon the point D: and AB upon DE; it follows from the equality of the angles that AC will take the direction of DF. But if the angles be unequal, although the point A be upon the point D; and AB upon DE; yet AC cannot poſſibly fall upon DF; becauſe AC will then take a direction, on the one ſide or the other of DF, according as the angle BAC is greater or leſs than EDF.

Laſtly let us ſuppoſe the two extremities of two ſtraight lines to be the ſame; may we not conclude from this that the lines are equal, and in the ſame direction, that is the one line will apply itſelf to the other; becauſe if it did not the two ſtraight lines would incloſe a ſpace. Therefore in this figure of ours, if it can be proved that the point B applies itſelf to E, at the ſame time that the point C applies itſelf to F; we may certainly conclude that BC will apply itſelf to EF and be equal to it.

Each of theſe ſuppoſitions ought to be examined frequently, and the conſequences which follow from them are to be ſtrongly impreſſed upon the memory; ſo that the very mention of the ſuppoſition may ſuggeſt the conſequence.

And here it may be proper to inform the ſtudent, that he is not to conſider this as any acquiſition of ſcientific knowledge; but only the conſequences which common good ſenſe cannot fail to draw, whenever theſe things become the ſubject of conſideration; theſe remarks are introduced not as teaching any thing new, but only to fix the attention. However the combination of theſe, as in the fourth propoſition, will carry us a ſtep beyond the common ſenſe of mankind: for he who firſt found out that the two ſides and included angle fix every part of the triangle beyond a poſſibility of change, had certainly more than a common notion of a triangle, and merited the high title of an *inventor*. The learner is now to endeavour to make himſelf maſter of the fourth propoſition, before he proceeds to the next chapter, in which he will

find

find ſome remarks tending to correct, confirm or enlarge the notions which he may derive from it according to the different degrees of attention beſtowed upon it.

CHAP. IV.

The ſame ſubject continued.

THE reader is without doubt ſurprized to find this affair of *ſuppoſitions* repreſented as a difficulty almoſt unſurmountable; and not a little inquiſitive, we may ſuppoſe, after the reaſons of this ſtrange *phænomenon*. Our own indolence is the real cauſe, for ſimple reaſoning is hardly ſufficient to ſet the mind in motion without ſome external application. When a problem is propoſed, there is ſomething to be done, and, which is very much to our purpoſe, may be done wrong: this ſets us upon thinking; and by giving a preference to one method above another, we come to diſtinguiſh between what is right and wrong, and become ſo much intereſted as to give the queſtion a ſerious examination. But the caſe is very different in theorems, for the conſequences are abſolutely fixt by the ſuppoſitions, ſo that a mind without experience of ſuch ſubjects has nothing to engage its attention.

It would be abſurd to propoſe a problem of this kind: Suppoſe two triangles to have two ſides equal to two ſides, each to each; and the angles, contained by the equal ſides, equal; it is required to make the third ſide, equal to the third ſide; and the two remaining angles equal to the two remaining angles, each to each; namely thoſe under which the equal ſides are extended. Here there is no room for a conſtruction; becauſe according to the ſuppoſition, the ſides and angles cannot be otherwiſe than equal. However it is neceſſary to ſhew that theſe magnitudes are really equal; and the ſuppoſition alone is what we muſt truſt to, for bringing this about: and we are at a loſs becauſe, for this end, we begin to derive the conſequences from the ſuppoſition before we have examined the different circumſtances of which it conſiſts, or without underſtanding what isimplied in the ſuppoſition, when it reaches beyond the obvious meaning of the words in which it is expreſt;

expreſt; or when *more is meant than* directly *meets the ear.* The mind muſt have room to exerciſe itſelf in, as well as the body; and it is rendered almoſt inactive by being confined to a ſingle ſuppoſition. But if the ſtudent has patience this may be remedied by varying the ſuppoſition; and then you may have all the latitude and range of thought, which is allowed in the conſtruction of a problem.

Let us ſuppoſe then that only AB is equal to DE, and the angle BAC equal to the angle EDF; but that we are quite in the dark as to AC and DF.

When the point A is put upon D, and AB upon DE the point B will apply itſelf to E as before and AC will take the direction of DF; but it is impoſſible to determine any thing concerning the poſition of the point C, only that it will be found ſome where in the line DF or in DF produced.

Again let us ſuppoſe only the two ſides equal; then the point A being put upon D; and AB upon DE; the point B will as before apply itſelf to E, but we can make no uſe of the other part of the ſuppoſition; for we can determine nothing concerning the direction which AC may take, becauſe nothing is ſaid concerning the angles or inclinations of the lines.

But farther, let us ſuppoſe, that not only AB is equal to DE and AC to DF and the angle BAC equal to the angle EDF; but moreover, that AB is equal to AC and conſequently DE to DF: now by placing the triangles as in the propoſition, we ſhall find the angle ABC equal to DEF; and not only that, but, by putting A upon D, and AB upon DF the point B will now apply itſelf to F becauſe AB and DF are equal by the ſuppoſition; and alſo the angle ABC will be equal to the angle DFE: but ABC has been already proved equal to DEF; therefore by the firſt common notion the angle DEF is equal to DFE: that is the angles at the baſe of the triangle DEF are equal: But this concluſion does not depend entirely upon DE's being equal to DF; but it is required beſides that BA ſhould be equal to AC and to each of the lines DE, DF and likewiſe the angle BAC equal to EDF. So that this caſe can by no means be conſidered as including the fifth propoſition, in which there is no other ſuppoſition but the equality of

of the two ſides; and as we muſt have an angle equal to the angle at the vertex of the iſoſceles triangle, this angle muſt be made, which, in our preſent circumſtances, will not be found ſuch an eaſy matter. And ſo much for hypotheſes or ſuppoſitions.

CHAP. V.

Containing a critical examination of the fourth propoſition.

OBJECTIONS are brought againſt this propoſition, as if the demonſtration proceeded by a mechanical application of the two triangles to one another. But whoever ſtarts ſuch an objection as this, has not the moſt diſtant conception of the demonſtration; or elſe he muſt be ignorant of the nature of a mechanical application. And a particular examination will put this beyond all doubt.

Suppoſe two triangles of braſs, made as accurately as poſſible, all their different parts correſponding exactly. When we apply them to one another, all that we can ſay is, that as far as our ſenſes can judge, the parts ſeem to agree. Now what knowledge is got from this? Certainly by this application, no property of a triangle can be diſcovered; we may form a concluſion concerning the accuracy and neatneſs of the workmanſhip; but nothing farther. It is impoſſible to ſay from this that any thing is equal, but what is made equal; no doubt a very curious diſcovery and tending greatly to the enlarging the boundaries of the ſcience!

The miſtake here ariſes from not conſidering that it is impoſſible to make the parts of the figure which are to repreſent the ſuppoſition unleſs you make at the ſame time the parts expreſſing the conſequences which, it is ſaid, will follow. The parts repreſenting the ſuppoſition are to be tried and examined by the eye, as well as thoſe which repreſent the conſequences; ſo that if any conſequence follow it muſt follow from nothing. And the moſt attentive ſtudent will retire from this contemplation, with a very curious piece of information; namely, that he had ſeen two braſs figures ſo contrived as to fit each other exactly.

We

We now proceed to confider the application made ufe of by Euclid in this fourth propofition. And firft it is to be obferved that his triangles are required to have no particular pofition; therefore the point A may be fuppofed to be upon D as well as any where elfe; and likewife the line AB upon DE; and all this might happen to any two triangles not determined to any particular pofition. Then AB being to DE the point B muft apply itfelf to E, it is impoffible to conceive it to have another fituation, which is very different from faying that as far as we can judge by our fenfes it feems to be fo; again if the angle BAC be equal to EDF, the line AC muft take the direction of DF; and it is impoffible to conceive it to take any other; and fo likewife the point C muft apply itfelf to F, whenever the line AC is equal to DF: and fo on to the coincidences of the different parts of the triangles; and all this is as different from a mechanical application, as light is from darknefs. For fuppofing what is taken for granted, it is impoffible to conceive the confequences not to follow; and this is certainly fcience if there be any fuch thing in the world.

But again it is alledged, why may not this method of applying be extended ftill farther by making it an inftrument for conftructing problems? as for inftance in the fecond propofition; why may we not fuppofe the line BC fo placed that B may be upon A, and the thing required is done? For a very good reafon, becaufe BC and the point A have each of them a fixt pofition already.

Upon the whole then we may conclude, that this method of application is perfectly fcientifical; and that whenever two triangles agree in the circumftances mentioned in this fuppofition; the confequences will always neceffarily follow; the bafes will be equal and the remaining angles, each to each, viz. thofe under which the equal fides are extended: Becaufe no confequence is deduced from the lines having any particular length, but only from their being equal: nor is it fuppofed that the angle contained by the fides is any particular angle, but only that the angles are equal. And this may fuffice for an anfwer to the objections commonly brought againft the demonftration of this propofition.

CHAP. VI.

Containing an explanation of the fifth propofition.

ONCE upon a time a certain father refolving not to be impofed upon by reports, determined to examine into his fon's progrefs in this fcience, produced the book and required him to demonftrate a propofition to which he referred: the young man though unacquainted with the fubject, taking courage from his father's ignorance, began very impudently in fome fuch manner as follows; Becaufe the angle ABC is equal to the angle CBA, therefore the angle DEF is equal to the angle CEF &c &c; ringing the changes upon fides and angles, until he had fpun out his demonftration to a decent length: and then kept filence in expectation of his fathers's opinion; who with a grave and important countenance remarked, "This is what we call demonftration."

Every one is fenfible, that it is contrary to common fenfe to imagine that the letters of the alphabet thus repeated can have any meaning; but the indolent reader ought to be informed, that the repeating fuch phrafes, in a regular order, mends the matter but very little, unlefs they convey to the mind their proper meaning: and unlefs the fourth propofition be well underftood, the moft of that which follows will be nothing but an infignificant jargon. As the whole fcience therefore depends fo much upon an accurate and comprehenfive view of this propofition, it would be proper for the learner, before he proceeds farther to take the opinion of fome acquaintance fkilled in thefe matters; who, by a particular examination might be able to determine, how far he can be properly faid to underftand it; and this friend is authorized, upon his failure in any point, to admonifh him, by faying, "this is "what we call demonftration."

I myfelf can trace every miftake concerning the following propofitions, or partial conception of their meaning, up to my ignorance of the full import and meaning of this propofition. For it is by no means to be underftood as applicable only to fuch triangles, as one may make ufe of to affift the imagination in tracing the

the ſteps of the demonſtration; but as carrying with it this extenſive and general meaning, that all triangles which have two ſides equal to two ſides, each to each, and the angles, contained by thoſe ſides, equal; that all ſuch triangles, I ſay, have their baſes equal; and the two remaining angles in every triangle, equal, each to each; viz. thoſe angles under which the equal ſides are extended. Or more properly that all ſuch triangles, whatever be their number, are but one and the ſame triangle. Or otherwiſe that the two ſides, and the angle contained between them, fixes every part of the triangle, beyond a poſſibility of change. But I ſhall now proceed to conſider the fifth propoſition. And the reader may recollect an attempt in the end of the fourth chapter, to derive this propoſition from the fourth, or rather to make it only a particular caſe of that propoſition, when the two ſides of each triangle are equal to one another; and I there gave the reaſons why it could not be conſidered as ſuch: neverthelefs the artifice of this propoſition will be the better underſtood by proſecuting that ſcheme a little farther, producing the equal ſides, and proving the angles below the baſes of ſuch triangles equal, in the ſame manner as the equality of thoſe above the baſe was there demonſtrated.

It was then obſerved that the ſuppoſition was too complicated for this purpoſe; becauſe not only DE and DF were to be equal to each other and to AB and AC, for this might have been allowed; but alſo the angles between the ſides were to be equal, which could by no means be allowed; becauſe according to the fifth, only one angle being given, the other was to be made equal to it; which is not at preſent poſſible.

If the reader has proſecuted this ſpeculation far enough, he will certainly admire the ingenuity of our author for his contrivance to make theſe angles equal, in a very elegant manner indeed: and which, it is curious to obſerve, neceſſarily requires that the ſides of his two triangles ſhould be unequal, For he makes this very angle which we are conſidering common to both, by making the ſides of the triangles take the ſame direction, the ſhorter ſide of the one being upon the longer ſide of the other.

But now it will be proper to turn to the demonſtration itſelf; and after a careful examination of its different parts, I would re-

commend a particular attention to the uſe which might be made of it, in order to impreſs the laſt propoſition more ſtrongly upon the memory. The great advantage of problems above theorems, for fixing the attention of the learner, has been mentioned already: and here a little of that advantage may be gained, by the particular view of the fourth exhibited in this propoſition; which will teach us how to repreſent by an actual conſtruction, the ſuppoſitions in the laſt propoſition: becauſe, if we draw two undetermined ſtraight lines, making any angle; we can cut off AB equal to AC; and AF to AG; and, by joining BG and FC form two triangles, with all the parts of the ſuppoſition in the laſt, accurately repreſented; and this not in imagination, but conſtructed by ourſelves; and the conſequences, not made, as was objected to the mechanical triangles, but left to follow from the conſtruction; the accuracy of which, the ſlow of imagination may examine by inſtruments; and thus get a kind of palpable evidence for the truth of the propoſition; and by varying the inclination of the lines, and the length of the ſides, climb up by degrees to ſomething like a ſcientific conception of its meaning.

The other two triangles FBC and CGB, though they repreſent a particular inſtance of the fourth propoſition, are not ſo fit for this purpoſe, not being ſo general as the firſt two triangles nor ſo much in our power, or rather indeed they are not at all in our power: We may vary the angle and the two ſides in the triangle ABG at pleaſure; but we have no power over the ſides, or angle BFC of the triangle FBC; for they become ſuch as may happen from the conſtruction of the firſt triangle: therefore the epithet *any* could not be ſo properly applied to the triangle FBC as to ABG.

I have recommended to thoſe whoſe imaginations are ſlow or inactive, theſe conſtructions and inſtrumental proofs of the ſeveral concluſions which follow from the ſuppoſitions in the fourth propoſition. And I now, rather more earneſtly, recommend the ſame to thoſe whoſe imaginations are diſpoſed to out run their judgement, leſt they ſhould ſnatch the concluſions without the premiſes.

I have only one remark more to make upon this propoſition, which the judicious reader has already made for himſelf, namely, how little there is to attend to, in this very formidable propoſition, this

this *pons asinorum*, upon a supposition that the fourth is perfectly understood: for except the circumstances of taking the equal angles CBG, BCF from the equal angles ABG, ACF; it has nothing peculiar to itself; every thing else consisting of particular prospects of the fourth proposition.

But if any one is desirous to have some kind of reason for fixing his attention a little longer upon this proposition, by taking a somewhat different view of the subject; let him suppose, instead of this fifth proposition, it is required to demonstrate that all the angles of an equilateral triangle are equal; for instance suppose the angles of the equilateral triangle ACB described in the first proposition. Produce CA, and CB; and the same construction and demonstration made use of in this fifth, will prove that the angle CAB is equal to CBA; and by producing AB and AC; in the same manner it may be proved that the angle ABC is equal to ACB; but it has been already proved that ABC is equal to CAB therefore, by the first common notion, the angle CAB is equal to ACB &c.

I shall conclude this chapter by desiring the reader to take every opportunity of correcting a prejudice, which it will require all his art to remove. We cannot help drawing consequences from the very position which the lines accidentally happen to take in that particular figure which we reason upon, though this particular position make no part of the supposition. The student may convince himself of this, if he read the fourth proposition by the assistance of the figure which belongs to it; and then again making use of the two triangles ABG and ACF in the figure to proposition fifth; making the same supposition in both cases. If the consequences are suggested to his mind more readily by one set of figures than another; this can arise from nothing but the stress laid upon the magnitude and situation of the triangles, which, by the very supposition have no particular magnitude or situation. But this prejudice will be lessened, and gradually removed by varying the position of the figures, turning them upside down; and by such other methods as will readily occur to the attentive reader; but above all by enlarging the figures, setting the imagination to work until they cease to be the objects of our senses. The center of the moon, when

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in the equinoctial, joined by ſtraight lines to the two poles of the Earth, would make a very proper iſoſceles triangle with which the ſtudent might finiſh his ſpeculations upon this propoſition.

CHAP. VII.

Concerning indirect demonſtrations.

INDIRECT demonſtrations are generally uſed, when the propoſition is the converſe of ſome other which has been already demonſtrated. A propoſition is ſaid to be the converſe of another, when the hypotheſis or ſuppoſition in the one is the thing to be demonſtrated in the other, and the contrary; thus the converſe of the fourth would be; ſuppoſing the baſe BC equal to the baſe EF; and the angle ABC equal to the angle DFE; as alſo, the angle ACB equal to the angle DFE; that then the two ſides BA and AC will be equal to the two ſides ED and DF, each to each; viz. thoſe ſides which are extended under the equal angles, that is AB equal to DE and AC to DF; as alſo the angle BAC contained by the two ſides equal to EDF.

Now this propoſition admits of a direct demonſtration; for if the point B be put upon E and BC upon EF, the point C will apply itſelf to F, becauſe BC is equal to EF; and BC applying itſelf to EF; alſo BA will apply itſelf to ED, becauſe the angle ABC is equal to DEF. Certainly for the ſame reaſon CA will apply itſelf to FD, that is, becauſe the angle ACB is equal to DFE: the point A will therefore apply itſelf to D; and ſo AB will be equal to DE and AC to DF and the angle BAC will be equal to EDF. &c.

But if we ſuppoſe the angles at B and C, not only equal to thoſe at E and F, but alſo equal to one another; we may prove AB to be equal to DE as before; and then by putting B upon F and BC upon FE we can prove that C will apply itſelf to E; and that the line AB will be equal to DF: but it has been demonſtrated that AB is equal to DE, therefore by the firſt common notion DE is equal to DF; and no doubt ſome authors would produce this as a demon-

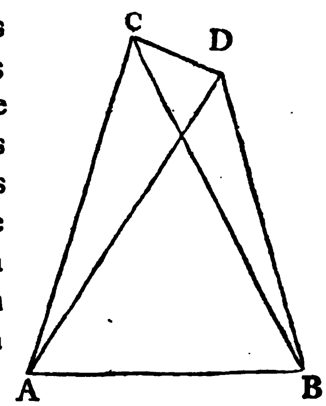
a demonſtration of the ſixth propoſition; but it is liable to the ſame objections, as it converſe, which have been mentioned already.

But inſtead of going any ſuch way as this to work Euclid has given a very elegant demonſtration of the ſixth propoſition, which is the converſe of the fifth; which the reader may pleaſe to turn to and examine; after which he may proceed with the following remarks.

And firſt it will be worth his while to obſerve the ingenuity which Euclid has diſplayed, in making the angles which are equal by the ſuppoſition, the angles between the equal ſides of his two triangles; and BC being a ſide common to both it remains only for him to make BD equal to AC; and to join DC. And here it may be aſked, and it is alſo the moſt important queſtion which can be put with regard to this propoſition, how comes it to paſs that he cannot find by his conſtruction that the points D and A are the ſame? It is true ſuch an inſtrument as the compaſſes would find out this if it diſcovered any thing; but this uſe of the inſtrument is rejected for reaſons given already. For the ſolution of this difficulty therefore the ſtudent may pleaſe to turn to the third propoſition, and conſider the apparatus made uſe of, for cutting off, a part from a line equal to another: and he will find that the ſuppoſing the problem poſſible implies that the laſt deſcribed circle muſt cut AB that is that the point D muſt be between A and B. And when we find that this ſuppoſition leads to an abſurdity what are we then to conclude, but that the problem is really impoſſible; which never can be if AB is greater than AC; therefore it is not greater. And I appeal to any one, whether the demonſtration is either difficult, obſcure or inconcluſive when it is conſidered in this manner. Nothing can be ſo ſenſeleſs as the objections uſually made to indirect demonſtrations. Every demonſtration may be loaded with objections until it becomes indirect: the reader will find a ſpecimen of that kind in my remarks upon the firſt propoſition; where a caviller is introduced denying that the equilateral triangle there deſcribed is a fixt magnitude. But our author never carries his reaſoning farther than good ſenſe requires; ſo that it is only ſuch propoſitions as this ſixth, which take an indirect form in his hands. And this kind of reaſoning will not apply unleſs we know how

how many ways the thing may be; as in the present instance AB must be equal to AC or longer or shorter: or else we must get at such direct consequences from one part of the supposition, as will overturn the other: as in the next proposition, which I must now beg the reader to peruse: after which I shall be ready to lend him my assistance in removing such difficulties as usually lie in the way of beginners.

It is supposed here, that AC is equal to AD; and at the same time BC equal to BD. The learner will do well to fix his attention entirely upon this: first tracing the consequences which follow from the supposition of AC's being equal to AD as far as they will go; which he may easily do; as it is only that the angle ACD is equal to ADC.



He is next to observe the consequences which follow from the position of the lines; and these consequences are, that the angle ACD is greater than BCD, which follows from this common notion that the whole is greater than its part; and also that the angle BDC is greater than the angle ADC, which follows from the same principle: and with a distinct impression of these things upon his mind, it is impossible to miss the conclusion that the angle BCD is much less than BDC: and this point being once gained; he is next to turn his thoughts to the other part of the supposition; in which it is pretended that BC is, also at the same time, equal to BD: and the consequence which follows from this part of the supposition cannot fail to engage his attention; being no other than this, the angle BCD is therefore, by the fifth proposition, equal to BDC: When we suppose AC equal to AD we must conclude that the angle BCD is much less than BDC; but these very angles must be equal upon the supposition that BC is equal to BD; is not this saying in the strongest terms that the two suppositions are inconsistent with one another; that is, that it is impossible for AC to be equal to AD; and at the same time, BC to be equal to BD. Which was to be demonstrated.

It

It will be a very neceſſary and uſeful taſk for the reader to ſet himſelf after he has finiſhed every demonſtration; to examine whether ſome conſequence has been drawn from every part of the ſuppoſition; for though it may not always be convenient for the author to draw them, yet the reader ſhould always do it for himſelf; for if the ſuppoſitions have no conſequences to the purpoſe either expreſſed or underſtood; they ought by all means to be omitted; and if the author truſted to the reader's ingenuity for *finding them out* he neglects his duty if this part of his buſineſs be overlooked. The neceſſity of this practice being thus made evident; I ſhall explain my meaning by a particular example; it is ſaid that AC and AD are to have the ſame extremity; but no uſe is made of this expreſly in the demonſtration: and yet without this the demonſtration could not proceed; becauſe ACD could not be a triangle unleſs theſe lines had the ſame extremity: neither could it be a triangle unleſs C and D were different points. It is moreover ſaid that they are to be towards the ſame parts: now if the points C and D were on different ſides of the line AB; the angles ACD and BCD would either be two diſtinct angles and the one not a part of the other; or if BCD be a part of ACD, then ADC would not be a part of BDC; which is abſolutely neceſſary for bringing out the concluſion which we aim at. Laſtly it is ſaid that the equal lines are to be terminated in the ſame extremity; and without this neither ACD nor BCD would be iſoſceles triangles; and then no inconſiſtency could follow, reaſon as long as we pleaſed. All this will be obvious by taking the points C and D on different ſides of AB, and joining CA, CB; DA, DB and CD; ſo that CD may cut AB or fall beyond the point B. And indeed it is very difficult to underſtand any general propoſition, without ſome repreſentation of all the different poſitions which the lines can take: and the reader cannot finiſh this propoſition better than by ſuppoſing the point D within the triangle ACB, and to aſſiſt his imagination, by making AC and AD equal with a pair of compaſſes; his demonſtration will reſt upon the ſame principles; only producing AC and AD; he is to reaſon upon the angles below the baſe; but every thing elſe will be the ſame as in Euclid's demonſtration.

I ſhall conclude this chapter with my opinion of indirect demonſtrations, which ſome very ignorantly object againſt. It would be abſurd to ſpeak of the certainty of a demonſtration: as to their uſe therefore, conſidered as an exerciſe to the mind, I think they are preferable, for ſeveral reaſons to direct demonſtrations. Becauſe in a direct demonſtration, there is generally ſome circumſtance which catches the ſenſes; and not being too difficult of perſuaſion, where our own eaſe is ſo much concerned, we are diſpoſed to reſt very well ſatisfied with that, though it may convey, but a very imperfect notion of the full extent of the demonſtration. But the contrary happens in ſuch demonſtrations as are indirect; for the reaſoning being generally at variance with the ſenſible repreſentations of the magnitudes, it requires ſome effort of the mind to get beyond the prejudice of ſenſe, ſo as to comprehend the force of the reaſoning: and when we cannot conquer our indolence or command our attention, it is a very decent excuſe to lay the blame upon the nature of the demonſtration. But I have been ſo particular in my examination of theſe two propoſitions, that the attentive reader can hardly want any farther information upon this head.

CHAP. VIII.

Of geometrical demonſtration.

IT will now be proper for the reader to endeavour, from a careful examination of theſe firſt ſeven propoſitions, to collect as accurate a notion as poſſible of a demonſtration, according to the rules which Euclid has preſcribed to himſelf.

We are apt to ſatisfy ourſelves of the truth of things, in the eaſieſt and ſhorteſt way we can; and even when we chuſe to conſider the properties of magnitude, inſtead of having recourſe to ſome general principles, we truſt to mechanical inſtruments; and very often to vague conceptions collected at random from accidental obſervation. But our outhor's plan is very different from this: The definitions, poſtulates and common notions, are the only foundation upon which his geometrical reaſonings are founded;

and

and when he mentions any thing as a property of a figure, he considers it as incumbent upon him to shew that it follows from or agrees with the definition of that figure; and this he does by the assistance of some common notion, postulate or other definition. And whatever property is not reducible in this manner, he considers as impossible to be demonstrated; though it might be a very obvious truth from other principles. Truth therefore is not so much his object as consistency; it does not appear directly to be any part of his business to collect as many useful facts as possible concerning the properties of figures; but only to be convinced that what he does produce have a solid foundation, making every part acquire additional strength by the consistency of the whole. He will never allow it to be said that such a property is so simple as not to require a demonstration; for to alledge this as a reason for taking any thing for granted, would be, upon his plan the greatest absurdity. For it may be said, if it be so very plain and obvious, whence does it derive this obviousness? If it be the property of a figure; that figure is defined to be so and so. How does it appear that it is consistent with that definition? If you point out the consistency, you have demonstrated the proposition; but if that cannot be done, it may be true according to your principles, but mine lead me absolutely to reject it.

That this is Euclid's plan will appear from the whole of his work, even to the astonishment of the attentive and judicious reader; when it is considered how his invention must have been perplexed, by the obvious, but inaccurate experimental tests, which would for ever be presenting themselves to his imagination; and how his patience would be tired out by the number of steps, which he forsaw to be necessary, before he could arrive regularly, at some conclusions seemingly very simple indeed. A man would feel him- /w self in a very askward situation, who should set about measuring magnitudes, without being able to judge when one straight line was longer than another; and yet it is wonderful, by how many steps, this very test is removed from the first principles; and which are all necessary for giving it a scientific stability. We find this delivered in the nineteenth proposition of the first book; but as this is rather foreign to my present purpose, and beyond the sup-

pofed knowledge of the reader, it will be fufficient to have mentioned it by the bye, only as a fact to illuftrate my meaning. I fhall therefore now beg the attention of the ftudent, while I review the firft feven propofitions, in fupport of my opinion concerning Euclid's plan of demonftation.

In the firft propofition his principles are the third poftulate; the firft poftulate; the fifteenth definition; the firft common notion; and laftly the twenty fourth definition. He is not fatisfied to make an equilateral triangle any how; but he makes it by the affiftance of his own inftruments and principles. The fecond propofition is conftructed and demonftrated by the firft poftulate, the firft propofition, the third poftulate, the fifteenth definition and the third common notion; and we have alfo here a proof how tenacious our author is of his principles; for it is not abfolutely neceffary that in this fecond propofition, an equilateral triangle fhould be defcribed upon AB; becaufe an ifofceles triangle would have anfwered the purpofe; for no confequence is drawn from AB's being equal to AD; as it would be fufficient if AD and DB were equal; but he could not make an ifofceles triangle, confiftent with his own plan and principles. The third propofition refts upon the fame foundation; its demonftration being fupported by the fecond propofition, and the third poftulate.

In the fourth propofition the principles which we reafon from, befides the fuppofition, are the eighth and twelfth common notions; to which every part of the demonftration may be reduced; for the application of the triangles to each other muft be allowed to be a proper principle to reafon from, otherwife it is difficult to conceive what ufe can be made of this eighth common notion, which is the definition of equality; it will not matter much whether this putting of the triangles be reckoned a conftruction or not; I am rather inclinable to confider it as a thing of its own kind, and not to be underftood either as an hypothefis or conftruction: I know Euclid makes ufe of the expreffion the point being put, which feems to imply a kind of conftruction; and yet it would be doing no great violence to the language, to paraphrafe it in this manner; the triangles have no particular pofition; that is they have no relation to any points, lines or angles, which can be affected

affected by our giving them a particular position; let us suppose them then to have such a position, as it is necessary magnitudes should have before that eighth common notion can be of any use; that is let us suppose their situation such, that the one is applied to the other; and the point A put upon D &c.

It is plain that Euclid himself found it impracticable to give this principle a more scientific form; and therefore instead of making a definition of it, left it among the common notions; but whatever opinion we adopt; it will answer our purpose, because every thing is inferred from the principles, which are laid down as such. If we were required to place triangles upon each other, which had a determinate position before, I would certainly consider it as a construction; but in the present case, I consider it only as a necessary apparatus before any consequence can be drawn from the eighth common notion. And as to the objections, upon a conceit that this is a mechanical application, they have been very fully answered already; so that upon the whole there is nothing in this proposition, but what agrees with the notion of a demonstration which has been delivered in the beginning of this chapter.

The fifth proposition seems by the references to depend upon the four first propositions and the third common notion; but as the construction required is only the simplest case of the third, the first and second propositions are not necessary for cutting off AG equal to AF. But the sixth requires the four first propositions; because the simplest case of the third, will not be sufficient for cutting off BD equal to AC; and, what I must beg the reader particularly to attend to, neither can it be done by a ruler and compasses; because there is something singular in this construction: for it is here required to do, what we afterwards find impossible to be done; and the unlucky circumstance is, that whoever trusts to a ruler and compasses will come to the knowledge of this too soon; a pair of compasses lets him into the secret immediately, before his mind is prepared for it by any scientific information; and hence will arise a gross misconception of the author's meaning; because the reasoning must appear useless and unnecessary, being intended to discover to a man what he has found out already. In all such cases as this therefore our ruler and compasses are to be laid aside,

and

and fome method devifed which may be more confiftent with the nature of the poftulates as here applied : for by keeping ftrictly to the information wchih we get from them, we never can difcover that we have been attempting to perform what is impoffible, before we are led to the abfurdity of concluding the triangle ABC to be equal to DCB. I would recommend the performance of fuch conftructions as this, by the hand unguided by any inftrument; as this will favour the condition of the propofition, becaufe fuch a conftruction can make no difcovery before the proper time. Not that I would *fuppofe* it performed; as that might fofter prejudices, but rather reprefent every ftep of the conftruction; BC being joined already I would defcribe upon it an equilateral triangle producing the fides &c. as directed in the remarks on the third propofition, until the laft defcribed circle cut AB in fome fuch point as D; becaufe the whole conftruction will be neceffary to convince the reader that he has no fcientific notice of the impoffibility of the problem from the conftruction, but muft wait for it until he come to the abfurdity. And hence it appears that Euclid requires every part of this propofition to be referred to the firft principles; contrary to the fentiments of thofe fuperficial readers, who imagine that they have only to fuppofe a point D, and the thing is done.

The demonftration of the feventh is reducible to the fifth; and the ninth common notion. And thus it appears, that Euclid's intention is not to fhew that the propofition is true in any manner; but that it is immediately connected with, and depends upon his principles, or in other words that his aim is not to perfuade but to demonftrate.

CHAP. IX.

In which is explained the geometrical meaning of the words finite and infinite.

" THE human underftanding, *fays Bacon,* fhoots itfelf out, " and cannot reft; but ftill goes on though to no purpofe. Thus " 'tis inconceivable there fhould be any bounds to the univerfe; " yet it conftantly, and, as it were, neceffarily recurs, that there " muft

" must be something farther. So again it cannot be conceived how " eternity should have flowed to the present time: and there is the " like subtilty as to the infinite divisibility of lines, &c. all arising " from the weakness of human thought." And whoever chuses to indulge this *weakness of thought* will be put into a fine train by reading what *Locke* has advanced upon the idea of infinity. But it is the end of all mathematical learning, instead of lending any assistance to encourage this kind of dreaming, to root out such a disposition from the human mind; and instead of such fooleries, to give it something to employ itself upon suitable to the powers and faculties of a human soul. An understanding which has any force will always shake off such dreams; the indulging in which is the sign of a weak intellect. For the philosopher and the idiot agree perfectly in their notions of infinity; and the difference between them is only discovered when they come to a comparison of determinate things and quantities.

The geometrical notion of infinity, as far as it regards extension, is derived from the second postulate; let it be taken for granted that a straight line may be continued directly forward: The geometricians never trouble themselves with multiplying any assignable parts of extension, in order to be convinced that after they have advanced millions of miles, they are still as far from the end of an infinite line as they were at their first setting out.

The two straight lines DE, DF in the second proposition, are infinite in the full and proper geometrical sense; and so likewise are AD and AE in the fifth; and the third proposition may be applied to construct this, because AF is less than AE: but why is it less? Because AF is a finite and AE and infinite line: and this instance fully explains what is meant by an infinite line in this science; for it means only a line longer than any determinate line which they have occasion to take; and farther than this they give themselves no concern. In the ninth proposition AB and AC are infinite lines; and the point D is taken that we may have a fixt line AD and so in other instances.

I am now arrived at the conclusion of this dissertation, in which I have endeavoured to point out such circumstances, as the thoughtless reader is apt to overlook. If it should seem strange to any one to find

/e find ſuch a cerimonious introduction to a ſimple, well defined and demonſtrative ſcience, he will be pleaſed to obſerve ſome peculiarities which diſtinguiſh this ſubject remarkably from moſt others. It is true the language is plain and ſignificant; but the truths to be communicated are conveyed in too few words to engage the attention of thoſe who have been accuſtomed to the figurative language in which moſt other ſubjects are delivered; and where the reader, inſtead of having conſequences deduced from every word, carries ſomething with him if he attend to but one word of three. And for this reaſon I have been a little diffuſe and circumlocutory; that the tranſition, from the common form of ſpeaking, to this very concise and accurate method of expreſſion, might be made with greater eaſe by the reader, after ſome part of the ſubject has been explained to him in his own way. But I muſt beg leave to introduce the ſtudent to Euclid himſelf, and defer all future intercourſe with him until he has reached the end of the firſt book.

DISSERTATION III.

THE first book of Euclid's elements will neceſſarily ſuggeſt a great variety of reflexions to a judicious reader; he will perceive a particular method of arrangement, and a particular manner of demonſtration which makes that arrangement abſolutely neceſſary; with ſomething characteriſtic even in his way of conſtructing problems. But an indolent reader requires to be put in mind of each of theſe particulars, otherwiſe he will be diſpoſed to overlook them. Some remarks therefore upon each of theſe heads may be uſeful to put him into a proper train of thinking; and this ſhall be the ſubject of the preſent diſſertation.

CHAP. I.

In which Euclid's method of demonſtration is proved to be neceſſary contrary to the opinion of Clairaut.

" Qu'Euclide ſe donne la peine de démontrer, que deux cercles " qui ſe coupent n' ont pas le même centre, qu'un triangle ren- " fermé dans un autre, a la ſomme de ſes côtés plus petite que " celle des côtés du triangle dans lequel il eſt renfermé; on n'en " ſera pas ſurpris. Ce Géométre avoit à convaincre des Sophiſtes " obſtinés, qui ſe faiſoient gloire de ſe refuſer aux vérités les plus " évidentes: il falloit donc qu'alors la Géométrie eut, comme la " Logique, le ſecours des raiſonnemens en forme, pour fermer la " bouche à la chicane. Mais les choſes ont changé de face. Tout " raiſonnement qui tombe ſur ce que le bon ſens ſeul décide d' " avance, eſt aujourd'hui en pure perte, & n'eſt propre qu'à ob- " ſcurcir la vérité, & à dégoûter les Lecteurs."

This is a very ſingular opinion concerning the motives which led Euclid to that rigorous method of demonſtration which he has adopted: for we are here told that it was not choice, but the circumſtances of the times in which he lived, which brought him to write as he has done: his living among ſophiſts, drove him beyond the bounds of good ſenſe: for if he had been left to follow his own inclinations, we have here room to ſuppoſe that his demonſtrations would have appeared in a very different form, or in other words, that had he been a Frenchman and lived in the ſame polite and happy times as *Clairaut* he would have written juſt as he has done. The French would have taken his word for it, that two circles, which cut one another, could not have the ſame center; and that polite nation ſurely never would have contradicted him if he had ſaid the ſame thing of two Ellipſes: but one thing I am certain of, that, had they been acquainted with no other principles but ſuch as this author would have us appeal to, they never could have contradicted ſuch an aſſertion upon any good grounds.

Now what I mean to prove, in oppoſition to this doctrine, is, that Euclid is remarkable for adhering to the very principles which *Clairaut* thinks he himſelf has gone upon: and I ſhall now produce as many inſtances as I think the reader's patience can well endure in proof of this aſſertion, namely, that Euclid every where diſtinguiſhes himſelf by keeping clear of ſuch things, as good ſenſe would decide of itſelf before hand. Now as the method of theſe two authors is not ſimply different, but directly oppoſite the one to the other; *Clairaut* and I muſt aſſign a very different office or employment to this ſame good ſenſe, the preſuming to teach which, he ſays, anſwers no other end but to diſguſt the readers. But I am certain good ſenſe will never be either affronted or diſguſted to have any thing ſet in a better or ſtronger light than what it appeared in before; and I challange any one to produce an inſtance, where the thought is not improved and rendered more accurate by Euclid's reaſoning; and it muſt be a ſtrange kind of good ſenſe, which could reckon ſuch reaſoning, as he ſays *en pure perte*.

But as to his looking upon himſelf as bound to convince obſtinate ſophiſts, it is very obvious that he never went in the leaſt out of his way on their account; and guided himſelf by very different maxims,

maxims, as will appear to every one who confiders his manner of reafoning, which is to the laft degree inconfiftent with any fuch fuppofition, as that he had the leaft intention to combat fuch prejudices as the fophifts raife, when they have a mind to fport with the credulity and ignorance of the multitude. It is true he furnifhes us with the means of efcaping out of their hands, if a proper ufe be made of his principles; but it never once entered his thoughts to combat their filly objections, which never could be obtruded upon any one, who is not entirely ignorant of the nature of the fubject.

However, as a full proof of what I now affert, I would recommend it to thofe who may chufe to differ from me in opinion, to read *Proclus's* commentary, who fets himfelf in earneft about this hopeful tafk; and there, it is granted he will find reafonings, *qui tombe fur ce que le bon fens feul décide d'avance*; and I will allow that fuch reafonings are of no other ufe but to obfcure the truth, and difguft the readers. The difpute therefore is brought to this fhort iffue; let the reader perufe the firft book of Euclid and then read *Proclus's* commentary, obferving the difference between the two kinds of compofition; and I am certain he will acquit Euclid of any defign to convince obftinate fophifts; and moreover muft be perfuaded that he has not only omitted every thing which good fenfe can *intuitively* determine; but that he has alfo left feveral things for the reader to fupply which would not come eafily under his own precife notion of demonftration; for Horace's rule was never better nor more properly applied than by this author;

—— et quæ
Defperat tractata nitefcere poffe, relinquit.

Several inftances of which, taken from his firft book, I fhall now proceed to enumerate, not confidering them as overfights, but as proofs of a moft refined and accurate judgement.

And firft, to begin with that famous principle by which the meeting of two ftraight lines is determined: We may readily fuppofe that Euclid introduced the feventeenth propofition, with a view either to demonftrate or explain the eleventh common notion,

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as being the converſe of it. For from this propoſition we learn that whenever the two lines meet, the angles made by the cutting line, upon that ſide, are leſs than two right angles; but we can by no means infer from this that the lines will meet whenever the angles are leſs than two right angles. But again he has demonſtrated in the twenty eighth propoſition; that when theſe angles are equal to two right angles the lines will never meet. But he cannot, from this ſay, that this is the only caſe in which the lines will never meet: only thus far he may go, that the ſuppoſition, from which he infers that they will never meet, requires abſolutely that the angles ſhould be exactly equal to two right angles; and the leaſt deviation from this ſuppoſition will render his principles of no avail; as no conſequence can follow from them; for their abſolute equality to two right angles and nothing leſs, is neceſſary to prove that the lines will never meet. Now this is ſufficient at leaſt to ground a perſuaſion, that in all other caſes they will meet; eſpecially when it is conſidered that ſuch lines are inclined towards one another, as may be ſhewn by drawing a perpendicular to one of the lines; which will prove that they are approaching towards one another upon the ſide where the angles are leſs than two right angles: and going farther from one another upon the ſide where the angles are greater than two right angles; but the opinion thus acquired is not of the nature of a demonſtration. Nor could he even ſay that a line interſecting one of two parallels, would meet the other; and indeed if this could be ſhewn upon ſcientific principles, it would be a demonſtration of the thing in queſtion: but this would be taking the thirtieth propoſition for granted, the truth of which depends upon this very principle.

We may therefore ſuppoſe that ſuch things as theſe and numberleſs others of the ſame kind muſt have occurred to a mind ſo fruitful in expedients upon ſuch occaſions, and were all rejected by him as falling infinitely ſhort of his idea of demonſtration: wherefore without perplexing his reader with impotent attempts towards a demonſtration; he judged it more proper to cut ſhort this fruitleſs ſearch by placing this principle among the common notions; and as it appears from this inveſtigation that it was not placed here without reaſon; ſo it has a right to keep its place, until it can be ſhewn

ſhewn that ſome abſurdity can follow from the uſe of it. And upon this point I have only to add, that it is certain from the general tenour of his demonſtrations, that if he could have exhibited it without any ſcientific defect, but in the clumſy dreſs in which ſome authors have arrayed it, he would have rejected the concluſion with diſdain, and left it where we now find it. So that it is not a demonſtration of any kind which was his object, but ſuch an one as is both elegant and forcible.

But further, I have mentioned before, what ſeems to be an exceeding good general rule of critiſciſm in this ſcience, that we ought always to examine if ſome conſequence has been drawn from every part of the ſuppoſition or conſtruction. Now there are ſeveral inſtances where Euclid has laid no ſtreſs, even, upon the moſt material part of the ſuppoſition; which ſhews how little he minded the conviction of thoſe *obſtinate ſophiſts*, for whoſe ſake according to *Clairaut* he gave his elements ſuch a formal dreſs, to the great diſguſt of every Frenchman; becauſe here would have been ſo good a pretence for ſtanding ſtill, that no arguments could have perſwaded theſe ſophiſts to proceed while ſuch blemiſhes remained. But to come to particulars.

In the twenty ſecond propoſition, it is required to make a triangle of three given ſtraight lines, but with this neceſſary limitation; that any two of them muſt be greater than the third. However no uſe is made of this limitation in the demonſtation of the propoſition; and I am perſwaded that our author omitted to prove that the two circles will cut one another, which depends upon this limitation, becauſe this proof does not admit of ſuch a natural and elegant turn, as he had determined to give to every ſtep of his demonſtrations; there being no direct principle to refer to, by which the interſection of circles can be proved; that they have ſome ſpace in common is the principle from which their interſection is to be inferred, which is indeed a common notion but not one of thoſe, which he has ſelected to draw conſequences from; concluding I ſuppoſe that all rational men, could make as ready and extenſive an uſe of ſuch notions as theſe, as he could, and therefore, that it would be unfair in him, to appropriate to himſelf either the notions or their conſequences; or according to *Clairaut*, that

that ſuch reaſonings would only fall upon, what good ſenſe had decided before hand, in a manner full as ſatisfactory.

Again, there is an inſtance of the ſame kind, in the twenty fourth propoſition; where the moſt material part of the ſuppoſition is never once mentioned in the demonſtration; viz. that the angle BAC is greater than the angle EDF; conſequences are tacitly drawn from it, like the taking for granted that the circles will cut one another in the twenty ſecond; for without this part of the ſuppoſition the angle EGF would not be a part of DGF, nor, if the point F was within the triangle DEG would this angle EGF be a part of the angle below the baſe of the iſoſceles triangle DFG; which is abſolutely neceſſary in order to prove that the angle EFG is greater than EGF; or, in other words, it follows from this part of the ſuppoſition, that DG always falls upon the ſame ſide of DF which it is repreſented to be upon in the figure; but it would be difficult to give this *reaſon* ſuch an elegant turn as to entitle it to a place in one of Euclid's demonſtrations.

But the ſophiſts would ſtick at much leſs matters than theſe; they would require him to prove in the firſt propoſition, that the two circles would cut one another; and that their interſection is a point: and in the ſecond that there would always be ſuch points as G and L: alſo in the fifth that AF and AE were two ſuch lines, that a part could be cut off from the one equal to the other, or that AE is greater than AF: they would likewiſe find ſeveral parts of the ſuppoſition in the ſeventh from which no conſequences are drawn, which I have taken notice of before; in the twelfth propoſition they would require him, to prove that a circle deſcribed with C for the center, and through any point on the other ſide of AB, would cut the line AB; which would give occaſion to his making uſe of the infinity of the line from which no conſequence is drawn, as the demonſtration now ſtands: They might likewiſe aſk him, upon the corollary to the fifteenth propoſition, whether when two lines meet at a point they make angles at their meeting equal to four right angles: and in the thirtieth propoſition, by what principle he makes a ſtraight line paſs through three points, that is, takes it for granted that it may paſs through three points, viz. a point in each line; for he ought to have taken two, and to have

have proved by the eleventh common notion that there would always be a third: Nor would it be very impertinent of them to require him to prove that AH and FG will meet in the forty fourth proposition; and indeed in that instance I cannot help being of opinion that the construction would have been more in Euclid's manner if he had made GH equal to BA and then joining HA had proved that HA was parallel to GB by the thirty third proposition. In short if it had been his intention to write for the conviction of the sophists, his demonstrations would have had a very different form from that in which we now find them. And it will be easy for any one who has read even the first book with attention to add a great many more to the above catalogue of omissions.

But I hope nobody will so far mistake my meaning here as to suppose that I consider such questions as these, either as captious or improper. They are mentioned only to shew that Euclid had some plan by which he directed himself in his demonstrations, very different from the supposition that it was to convince obstinate sophists. For I am so far from considering such questions as improper; that I think they are such as every one must ask himself, and a great many to the same purpose, otherwise I am certain he will have a very imperfect notion of the book; and whoever is in earnest to understand this subject should take every opportunity of devising such objections; the proper answers to which will be readily suggested by what Euclid has said. But this should be set about upon some regular plan; and I would therefore recommend it to the student, after he has finished the first book to give it a second reading, examining every supposition and construction, and the consequences drawn from them; and their full importance in the proposition; which is not to be determined from the number of words in which they are expressed, but by the dependence which the conclusion has upon them: for I have an instance just now in my view, in which I may safely say that the first reading hardly ever discovers the full importance of the supposition: what I mean is the supposition in the forty seventh proposition; from which the consequence that follows is; CA and AG as also BA and AH make but one straight line. Now whoever has read this proposition without perceiving that the conclusion depends entirely

upon

upon this, cannot be ſaid to have underſtood the demonſtration; and that this is true appears; becauſe the ſquare BG is only double of the triangle BFC becauſe GAC is a ſtraight line. The reader is alſo carefully to obſerve whether the concluſions are general or particular; becauſe it may ſo happen that a particular poſition of lines or points will lead to a concluſion, which will by no means follow if theſe are changed; and often if it do hold good it may require ſome new ſteps to come at the concluſion. But it ſeems neceſſary to illuſtrate this by particular examples.

In the eleventh propoſition of this firſt book, it is required to draw a perpendicular to a line, from a point given in it. This may be divided into two caſes; for the point may be at ſome diſtance from the extremity, or it may be the extremity of the line. Euclid has conſidered the firſt caſe only: and there is but one falſe conſequence which could be drawn, by taking the other for granted; and what is very remarkable *Simſon* in his edition has hit upon this very conſequence, in attempting to prove that two ſtraight lines cannot have a common ſegment; becauſe it muſt be taken for granted that two ſtraight lines cannot have a common ſegment before a perpendicular can be drawn from the extremity of the line; for as the line muſt be produced, without this limitation, it may take two directions.

Again in the twenty fourth propoſition, the point F may have three different poſitions; it may be without the triangle DEG or in the line EG or laſtly within the triangle: For a particular inſtance the ſtudent may make ſuch a triangle as ABC in the figure to this propoſition; having AB longer than AC; and after the triangle DEG is made; with the center D and diſtance DG deſcribe a circle; which will cut EG; now any ſtraight line drawn to the circumference of this circle, from the point D, within the angle EDG, may repreſent the line DF; from which the three different poſitions will be obvious. When the point F falls within the triangle, the concluſion may be inferred from the twenty firſt propoſition; but it would be more uniform to produce DF and DG; and to reaſon upon the angles below the baſe, in the ſame manner as Euclid has done upon thoſe that are above it. It is true *Simſon* reduces this to one caſe, but I think not in the manner of Euclid;

for

for there is no circumſtance given by which we can determine which ſide is the longeſt; there ought therefore to be added to the preſent ſuppoſition, *AC being longer than AB*; but the determination of this point I ſhall leave to the curious. But farther I would even recommend the ſame kind of examination, where there was no probability of finding any variation; as for inſtance in the ſixteenth propoſition, the concluſion depends upon FCD's being an angle; for if that could ever ceaſe to be an angle, the outward angle might be equal to the inward oppoſite one; but that is impoſſible becauſe then the point F would be in BC produced; and conſequently the point E upon C that is the middle of a line at its extremity, which is abſurd.

Having been thus particular in ſhewing what is not Euclid's method of demonſtrating, I ſhall conclude this chapter with a few inſtances tending more directly to explain what it is.

And his great peculiarity ſeems to be a determined reſolution always to refer directly to ſome principle, and never truſt to a vague conception; or more properly never to make uſe of a vague expreſſion: The thirteenth propoſition is a remarkable inſtance to this purpoſe; That the angles are equal to two right angles ſtrikes the ſenſes immediately and produces a conviction which has ſomething fluctuating in the nature of it; ariſing from obſerving that the angle ABD is above a right angle, by juſt as much as ABC is leſs than one; now this is not a principle ſufficiently diſtinct to reaſon from in a demonſtrative ſcience; and Euclid has ſhewn great art in the demonſtration of this very propoſition; which he has reduced to the common notion, *magnitudes which are equal to the ſame are equal to one another*; and how accurately he keeps up, through the whole demonſtration, to the notion of the angles being magnitudes, cannot fail to engage the attention of a judicious reader. The twentieth propoſition furniſhes an inſtance to the ſame purpoſe: every one believes that two ſides of a triangle are greater than the third; and he may take up this opinion from a conſideration of many different circumſtances; he may conſider that a ſtraight line muſt ſurely be the ſhorteſt way between two points: or he may truſt to the judgement of the aſs of the *Epicureans*; when a bundle of hay is placed at one of the angles; but

ſtill he will find ſomething unſettled in his conviction as reſting upon no determinate principles; and Euclid never leaves any thing which he takes in hand to demonſtrate, in this unſettled ſtate: he makes this property follow, not from any unſtable principles or random conceptions, but from the very nature of a triangle; and a thorough examination, of the whole appartus neceſſary for the demonſtration of this principle, will give the intelligent reader a wonderful inſight into Euclid's method of demonſtration.

It ſeems almoſt a deſperate undertaking, to endeavour at reviving a taſte for accurate demonſtration; becauſe the common vague conceptions of magnitude, proped by a ruler and compaſſes, when they begin to totter, are judged capable of performing every thing that can be expected from this ſcience. But if a reformation is at all practicable, it ſeems moſt likely to be brought about, by turning the attention of the world to Euclid's method of demonſtration; and I ground my hopes of ſucceſs upon two circumſtances; namely that the modern writers of elements of Geometry neither underſtand the nature of their own demonſtrations, nor the force of thoſe of Euclid. Becauſe if they underſtood the force of Euclid's they would be aſhamed to dignify their own with ſuch a name; and if they underſtood the nature of their own, they muſt perceive that a great deal of expence both of time and thought might be ſaved by delivering their propoſitions as facts, which had been demonſtrated; unleſs indeed we ſuppoſe that what they call demonſtrations are added merely in compliance to vulgar prejudice. Now could a judicious reader, no ways intereſted in the iſſue of the debate, be once brought to weigh the merits of ſuch demonſtrations; he muſt either decide in favour of ſuch as Euclid's, or declare that all demonſtration was now become unneceſſary; and that therefore the farce of it ſhould be laid aſide. But upon ſuch an event, ſuch is the charms of truth, I would naturally expect a conſiderable number to leave their new maſters, as ſoon as they had relinquiſhed their pretenſions to demonſtration: and it is upon ſuch hopes only that I proceed to give ſome other inſtances as characteriſtical of Euclid's method of demonſtration.

Euclid's demonſtration of the ſeventh propoſition, may be compared with the one given in *Deſchales's* edition of Euclid; which

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was no doubt intended as an improvement in some respect or other. And this seems to me the more extraordinary because the demonstration of the seventh is remarkably distinct and pertinent; leading to the conclusion so directly that the full force of the demonstration must be perceived by the mind: and I am persuaded that our author would have given up all thoughts of writing his elements, if he had found himself obliged to rest such a material part of his science upon so unstable a foundation, as the description of two circles, with an indeterminate radius and even before it could be known in how many points their circumferences would cut each other; because every point of intersection, that did not lie on the other side of AB would contradict the proposition; and prove that two triangles might be placed according to the supposition, until these points were reduced to one, which is not so easy to do as a superficial reader may imagine. But there is nothing which I have observed Euclid more cautiously to shun than the describing a circle with an indeterminate radius; or more properly, than the fixing a line, otherwise undetermined, for the purpose of describing a circle. It is true he makes use of consequences which must be derived from circles described in this manner; but this is supposed to be transacted in such a manner as to bring no disgrace upon the science. In the sixteenth proposition AC the side of any triangle is required to be cut in halves, which cannot be done until the line is fixed; but then I immediately perceive that, notwithstanding this, as no consequence is deduced from AC's having any particular length, the demonstration is nevertheless general; so that I can finish my construction and have AC as undetermined as I found it; so that this transaction is supposed to be carried on in private by the reader himself; and which he acquiesces in, as bringing no reflexion upon the science.

There is another thing which I shall just observe before finishing this chapter: Euclid's demonstrations are often more general than they seem to be; for whenever, it would be only a repetition of the same steps, he always omits them, whether in a construction or demonstration. The forty fifth proposition furnishes a remarkable instance of this; he shews how to turn a four-sided figure into a parallelogram &c. and as no new circumstance would

would occur, whatever be the number of ſides of the figure; he concludes that it is general, without adding a word more; which has miſled ſome ignorant conceited people to think that they improve upon Euclid by ſhewing a ſimpler way of turning a four-ſided figure into a parallelogram; never conſidering that Euclid has turned any rectilineal figure into a parallelogram, whatever be the number of its ſides; and whatever be their poſition to one another. Now I hope it is evident that Euclid had ſome other plan in his demonſtrations than to convince the obſtinate ſophiſts; and what that plan is I ſhall explain in the next chapter.

CHAP. II.

Of the arrangement of Euclid's propoſitions.

THOSE who know nothing of geometry tell us the ſcience ſhould be delivered according to the following arrangement. We ſhould firſt begin with *a point*; and after having laid before the reader all its various properties, ſuch as poſition, want of dimenſions and the like, we are next to proceed to the ſtraight line; and after delivering every thing that can be ſaid of a ſingle ſtraight line, we may then advance a ſtep farther and take two of them under our conſideration; firſt enquiring into the properties of ſuch as are parallel; and this ſubject being exhauſted we may next examine two ſtraight lines that meet ſo as to form an angle; which will furniſh us with a fruitful ſource of ſpeculation; becauſe this will now be the proper place to treat of angles; to ſhew how they may be proved to be equal, and into how many kinds they ought to be divided, not forgetting to prove that a right angle conſiſts of ninety degrees. Now ſuppoſing the properties of ſtraight lines exhauſted, the natural tranſition is to ſuch as are crooked, arranging them into different orders, beginning with the circle; but take care not to conſider it as a figure; for that would deſtroy all order; for it is the circumference only that belongs to our preſent ſubject: we may next proceed to the conic ſections, and ſo on to other lines.

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The business of lines being finished, there would be abundant matter for dispute, when we came to treat of figures, whether we should begin with the triangle or the circle; with the triangle as being the simplest rectilineal figure; or with the circle as being bounded by one single line; the circle might probably have the preference upon a full examination as being a *perfect figure*, and for other reasons which it would be too tedious to insist upon at present. But supposing this difficulty got over; the triangle would certainly claim the next place; and of all the different kinds of triangles, the equilateral would put in for the first; the isosceles would deserve to be considered next; from the isosceles the transition must be to the scalene; taking care all the time, in delivering their properties to say nothing of their angles, for that would be to confound the nature of things; because they are afterwards to be divided according to their angles, right, acute and obtuse; and this new division would furnish the proper place for that part of the subject. From triangles the most natural progress would be to the square, and then to figures less regular. And this is the true philosophical arrangement.

Such are the engines which *Folly* has been erecting for battering this noble work of our author; seconded with other auxiliary schemes, the force of which seems leveled against it, though more indirectly. But what holds the greatest vogue at present, as falling in with our indolent disposition, and bids fair for driving him to the last extremity, is the proposal of considering no more of the subject than what is absolutely necessary; with the inviting title; *as applied to such and such useful purposes of life.* Elements of geometry carefully weeded of every proposition tending to demonstrate another; all lying so handy, that you may pick and chuse without ceremony. *This is useful in fortification*: you cannot play at billiards without this. You only look through *a telescope* like a *Hottentot* until this proposition is read; with many such powerful strokes of Rhetoric to the same purpose. And upon such terms, and with such inducements who would not be *a mathematician*? Who would go to work with all that *apparatus* which I have described as necessary for understanding Euclid; when he has only to take a pleasing walk with *Clairaut* upon the flowery banks of some delightful river,

river, and there see with his own eyes, that he must learn to draw a perpendicular, before he can tell how broad it is: such is the ingenuity of these Frenchmen,

> Ut puerorum ætas improvida ludificetur,
> Labrorum tenus.

I had formerly the ambition to join myself to this fraternity, and taking a walk in the *french stile* along the banks of a river, proposed to teach the reader, how to measure its breadth by his hat; but I soon involved him so much in the properties of triangles, that I dispaired of making my point good, in the true elegant and undemonstrative manner; and so was obliged to get into the old beaten road as fast as possible.

But there is another rank of geometricians, between those, I have been speaking of, and Euclid: whom for distinction sake I shall call the *omnipotent*. These setting out with the supposition, that they can do every thing, undertake to demonstrate every thing. An *omnipotent* geometrician makes nothing of demonstrating the fifth proposition, for it is only supposing that he has cut the angle at the vertex in halves and the thing is done; never considering that the doing of this, requires it to be first proved that the angles at the base of an isosceles triangle are equal to one another. The only inconvenience of this method is, that should the author's omnipotency be once called in question; his whole fabric must be leveled with the ground.

But to return to *Clairaut*; I would have nobody imagine that I entertain a mean opinion of his abilities, from any thing that has been said: we only happen to differ as to the proper manner of communicating mathematical knowledge; which I contend should be either as facts or as demonstrations; and that there is no middle way; being persuaded that the mixing a kind of theory with the practice has been of bad consequence to both; as this has been the occasion of putting very improper persons into employments, by which many fine opportunities of making improvements have been lost to the world.

But farther *Clairaut's* method is even unnatural; for the science never could have been invented according to his plan; because it

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impoſſible that human reaſon, unleſs unnaturally confined, would be kept floating upon the ſurface of things: it is of the nature of man, when left to himſelf, to ſeek a ſolid foundation. His book, it is true, will make a ſuperficial reaſoner; but mankind left to themſelves, if they did ſo much, would neceſſarily do a great deal more.

And I now proceed to give an account of what ſeems to me a much more natural courſe for them to have taken in proſecuting their diſcoveries in this ſcience; which will at the ſame time explain the nature of Euclid's arrangement, and obviate every objection which has been made to it: but this ſhall be the ſubject of the next chapter.

CHAP. III.

The ſame ſubject continued.

It will be found to be a miſtake to ſuppoſe that we naturally begin to examine the ſimpleſt things firſt; for things are neither ſaid nor done in the ſimpleſt manner by the firſt adventurers. Our attention is more like to be drawn to a ſubject by ſome particular circumſtances, which made the conſideration of it neceſſary, than by any determination of the mind to make it a matter of contemplation; and by coming upon us thus, as it were unawares, our underſtanding is overpowered, when we undertake the examination of a thing encumbered with ſo many additional parts, which have no connexion with the ſubject in queſtion; and ſuch as it will require leiſure, ſkill and experience to ſeparate from the others.

We are not therefore to imagine that the examination, either of *a line* or *a point* would be the firſt attempts for ſetling mens notions with regard to magnitude. It is more reaſonable to ſuppoſe that the firſt kind of magnitude which would fall under the conſideration of men, circumſtanced as we are in ſociety, would be rectilineal figures, and perhaps circles or even ſolids of ſome determinate ſhapes: not only becauſe nature produces ſuch a variety of them ready made to our hands, thus, as it were, inviting us to make them the ſubject of our mediation, but ſo ſoon as property in

in land became eſtabliſhed, a man had only to diſcover that a greater quantity of it would yield a greater increaſe, to engage him to pay attention to this quantity; either that he might be the better able to impoſe upon his neighbour, or to guard himſelf from the impoſitions of others.

And although they might at firſt imagine that the bounding lines would be ſufficient for aſcertaining their property; yet they would come at laſt by reaſoning and blundering, to perceive that the ſhape is neceſſary to be conſidered, as well as the bounding lines. This might lead them to make pictures or reſemblances of their fields, by way of aſcertaining their property, until by freſh experiments and blunders, it may be ſuppoſed they could perceive, that theſe repreſentations, which I imagine to be made by the eye, without the aſſiſtance of other principles or inſtruments than ſuch as occaſionally offered themſelves, were alſo inaccurate. Or even if this could have determined one's property, when joined to other circumſtances upon the ſpot; yet ſtill a very material defect would be obvious; as there would nevertheleſs be wanting, any thing like an accurate way of comparing different figures with one another, eſpecially when their ſides are numerous and their ſhapes unlike. And it is probable that they would at firſt drop all thought of being able to compare ſuch with any degree of accuracy, but giving them up as unmanageable, would bend their whole attention to the ſimpleſt kind, very probably to the triangle; until the great ſtep was taken, which was to lay open the whole myſtery of this ſcience, by the man who firſt diſcovered that all rectilineal figures may be divided into triangles; and this ſtep once taken, they would then begin to tread on ſcientific ground. They would be made ſenſible of the uſe of the triangle, and exhauſt all their thoughts and ingenuity to diſcover its properties.

But it ſhould be obſerved, becauſe I believe it to be the great ſpur to improvement, that there would probably appear great inequality in the accuracy of the different concluſions which their ingenuity would lead them to, when their curioſity was once awakened by ſo fair a proſpect of being able to maſter their ſubject: ſome amounting to abſolute demonſtration, while a great many things perhaps fell ſhort of ſuch an evidence as was neceſſary to perſuade

persuade them that the conclusions were true; and such discordant parts would necessarily awaken their industry to give the whole an uniform appearance. It would be seen that if such a thing were proved such another would follow; and they would collect as many of these consequences as they could relating to triangles; what would prove them to be equal or unequal; until they came to perceive the remarkable utility of this wonderful figure, and its aptitude to explain and connect the simplest and most intricate parts of the science.

And this seems to me to be the most natural account of the original and improvement of geometry; because I am persuaded that if any thing, *even* uniformly inaccurate, had been produced by themselves; or if the regular mechanical rules sufficient for the ordinary purposes of life had been communicated to them by some superiour Being; mankind would have never thought of any farther improvement; and the human race would have drudged on in the same beaten path void of curiosity and fully satisfied with their present acquisitions. But this we find to be a fact by experience; for those who are instructed only in the practical parts of this science, proceed with as little hopes or desire of improvement, as the very instruments with which they work.

But to return to the subject; let us suppose at first that they had it only in view to give irregular figures a more regular shape; or in other words, that they wanted to perform the problem, which Euclid has given in the forty fifth proposition of his first book; namely, to turn any rectilineal figure into a parallelogram having a given angle: it will be found that almost every proposition which goes before, is necessary towards a proper solution of this problem, and if the student would be at the pains to examine how far this is true in fact, it will assist him much in comprehending Euclid's arrangement; especially if he attend particularly to the use which is made of the triangle: and with a few remarks upon this I shall conclude the chapter.

And first it is to be observed that straight lines and angles are too simple of themselves to bear a strict examination, with any hopes of discovering their properties; and if we want to prove them to be either equal or unequal; the lines must be sides of a triangle

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and the angles are angles of ſome triangle; the fourth, fifth, ſixth, ſeventh and almoſt every propoſition in the book furniſhes us with inſtances to this purpoſe; and it is owing to this ſure guide deſerting us that, we are brought into ſuch difficulties by the eleventh common notion. It is true that by the aſſiſtance of a perpendicular, we ſeem to make a few ſteps in comparing angles which do not belong to any triangle, as in the thirteenth, fourteenth and fifteenth propoſitions, but as theſe depend upon the perpendicular, they are virtually ſupported by the triangle: and the ſame may be ſaid where parallel lines are concerned. That the lines from the center to the circumference of a circle are equal, is the only principle by which lines can be proved to be equal independent of the triangle. Secondly we may obſerve how the tranſition is made from triangles to fourſided figures; and the triangular ſpaces compared with parallelograms, by which a foundation is laid for turning all rectilineal figures, however irregular into parallelograms. And laſtly when the uſefulneſs of the triangle is conſidered, the reader will not be ſurprized to find ſo much pains taken, to ſettle every circumſtance which can prove them to be equal, and in what caſes ſome of their parts may be equal and the others not.

CHAP. IV.

Containing ſome remarks on the conſtructions in the firſt book.

IT has been remarked already that our author diſcovers great caution in the uſe of the poſtulates, by keeping all mechanical inſtruments as much as poſſible concealed from public view: and indeed he ſeems almoſt aſhamed of them, for which reaſon he never conſtructs his problems upon a ſuppoſition that you make uſe of any particular inſtruments; and this is neceſſary to be obſerved before we can form a proper judgement of the ſimplicity of his conſtructions. The common modern way of judging upon this point, proceeds upon a very mechanical principle indeed; for the ſimplicity of a conſtruction is to be determined by the number of times which you have occaſion to open your compaſſes: but as Euclid

Euclid knows nothing of a pair of compaſſes, it can never be ſuppoſed that this is the teſt to which he appeals; but to the nature and number of ideas ſuggeſted to the mind by the different ſteps of the conſtructions. He does not care how often you are obliged to open your compaſſes, provided the perceptions, conveyed to the mind by his conſtructions, be the moſt ſimple and general which the circumſtances of the caſe will admit of. Some particular inſtances will explain my meaning.

In the twenty third propoſition we are required to make one angle equal to another; and you ſave yourſelf the trouble of one opening of your compaſſes by making DEF an iſoſceles triangle: but Euclid requires you only to join EF; which certainly is a ſimpler conception, than when you are moreover required to take care that your other two ſides be equal; and has alſo this great ſcientific advantage, that it is more general, by ſhewing that any triangle will anſwer the purpoſe; when your mechanical conſtruction goes upon the ſuppoſition that a particular kind of triangle is neceſſary. Thus it often happens that our compaſſes and underſtandings are at variance; or rather our hand and head incline different ways; what is moſt intelligible to the head, the hand finds the greateſt difficulty in performing.

In the ſame manner, in the ninth propoſition, three circles muſt be deſcribed according to Euclid's plan, whereas mechanically, which ſuppoſes that we can deſcribe an iſoſceles triangle upon a given ſtraight line, one opening of the compaſſes will anſwer the purpoſe; and there is a like difference in the ſcientifical and mechanical conſtructions of the tenth, eleventh and twelfth propoſitions.

The drawing of parallel lines Euclid uniformly refers to one principle; and this problem is performed by making the alternate angles equal; and this the nature of the ſcience requires that he ſhould do: however I have no objection to the uſe of any inſtrument which may anſwer the purpoſe, provided the ſtudent know what he is doing, and do not work at random without any ſettled principles.

Parallel rulers are of all contrivances, the moſt inconvenient for this purpoſe of drawing parallel lines; but yet it might be of uſe

for the ſtudent to conſider upon what principles we pretend to uſe them: and he will find that they are conſtructed upon geometrical principles, though not by any propoſition in Euclid; for their parallelism depends upon the converſe of the thirty fourth propoſition, namely, that if the oppoſite ſides of a quadrilateral figure be equal, they will be parallel; the demonſtration of which will be obvious to the attentive reader; and he might at the ſame time conſider the converſe of the other part of the ſame propoſition, viz. that if the oppoſite angles of a quadrilateral figure be equal, the figure is a parallelogram; this follows directly from this conſideration; the oppoſite angles being equal to one another, and the four angles equal to four right angles, any two of them taken in order round the figure will be the half of the four right angles; that is equal to two right angles, and therefore the lines are parallel.

But the moſt convenient inſtrument for drawing parallel lines, as alſo a perpendicular, is a triangular piece of wood in the ſhape of a right angled triangle: for, by making this ſlide along the edge of a ruler, parallel lines may be drawn, upon the principle that the outward angle, is equal to the inward &c. by the twenty eight propoſition. And if you have two parallel lines given you, a parallelogram may be formed by the thirty third propoſition, by taking two equal parts of them, and joining their extremities which are towards the ſame parts; in ſhort I would recommend every kind of conſtruction which is derived from ſcientific principles, provided the reader give himſelf the trouble to find out the principles.

It ſometimes happens that a direct conſtruction will not anſwer the purpoſe; of which the amendment propoſed by me in the conſtruction of the forty fourth propoſition is an inſtance; and if any one try to demonſtrate the forty ſeventh propoſition, by deſcribing the ſquare upon the ſame ſide of BC upon which the triangle ABC ſtands; he will meet with another inſtance of the ſame kind; for if he deſcribe this ſquare, he will find it no eaſy matter to prove that its ſides are terminated in FG and KH or in theſe lines produced, which is abſolutely neceſſary to make good the demonſtration: but if he draw perpendiculars from the points B and C terminated in theſe lines and joining their terminations, it is eaſy

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to prove the figure to be a square; such indirect constructions should be particularly noted by the learner wherever they occur. In short to conclude what I have to say about constructions, I would have the student exercise himself, in them upon every occasion, or even to seek occasions for doing it, only with the caution to beware of contracting such prejudices as might corrupt his scientific principles, which can never be the case, if he know what he is about, and consider the nature and extent of his instruments.

CHAP. V.

Containing such remarks as may enable the reader to make a proper estimate of his progress in geometry when he has made himself master of this first book.

BEFORE he proceeds to the second book it will be proper for the reader to take a review of this first, not as a learner, but in order to make a proper estimate of his geometrical acquisitions: and he will not be much out in his reckoning if he judge of the whole by what he knows of the triangle. I have mentioned already that this figure is our great instrument of discovery, being equally necessary for determining the simplest properties of straight lines, and angles, as those of the most irregular figures.

The first general property which we learn of this figure is that all triangles are equal, which have two sides equal to two sides each to each, and the angles contained by the two sides equal; and that this equality does not simply hold with respect to the spaces, but is true of every single part of the triangle; their bases are equal, the two remaining angles in each are equal, viz. those under which the equal sides are extended, each to each: or in other words that the two sides, and the angle between them, fixes every part of the triangle without a possibility of change. The fifth and sixth propositions contain more partial properties of the triangle; by proving that the equality of the two sides will always insure the equality of the two angles at the base: and conversly that the equality of the two angles at the base fixes the equality of the

the ſides. Thus we know what will prove the triangle to be iſoſceles; and we know the moſt remarkable conſequences which follow from its being ſuch a triangle.

In the eighth propoſition it is demonſtrated that all triangles which have their three ſides equal, each to each, are equal in every reſpect: or in other words, that the three ſides fixes the triangle, ſo that neither its ſpace or angles can poſſibly vary ſo long as the ſides continue the ſame: and in the twenty ſixth propoſition we have two other teſts to the ſame purpoſe: for if two angles and the ſide between them be given or fixt we cannot change any part of the triangle, neither the other two ſides nor the other angle; ſo that all triangles which have theſe equal muſt be equal in every reſpect: or ſo long as two angles of a triangle and a ſide extended under one of them are fixt the whole triangle is fixt; therefore all triangles which agree in theſe particulars muſt be the ſame or equal in every reſpect.

Theſe four teſts which fix the triangles abſolutely ſhould be carefully examined not only ſeparately, but compared together before the reader can make the proper uſe of them.

It is not only neceſſary to obſerve in what circumſtances the triangles may be demonſtrated to be equal; but it is of no leſs importance to ſettle in what caſes, we can ſay, that the whole triangles or certain parts of them muſt be unequal. And firſt it appears, from the ſeventh propoſition, that if the two triangles be upon the ſame baſe and upon the ſame ſide of it; their ſides which are terminated in the ſame point cannot be equal. Alſo it follows from the twenty firſt propoſition, that if the triangles be upon the ſame baſe, and the vertex of the one triangle within the other; their ſides cannot be equal, either each to each or taken both together; but that the ſides of the included triangle will be leſs than the two ſides of the other.

Such concluſions as theſe indeed, we are both very dexterous and confident in drawing at random; and the following would appear ſtrikingly concluſive; becauſe it is proved in the fourth propoſition, that, when the two ſides are equal, each to each, and alſo the angle contained by the two ſides, therefore the baſe is equal to the baſe; therefore undoubtedly we may infer from this, that

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that if the included angles are not equal, neither will the bases be equal, but the one must be longer than the other; and certainly the larger angle ought, by the rule of right, to have the longer base extended under it. But softly, good sir, this is not geometrical reasoning: for although the equality and coincidence of the bases is demonstrated in this proposition when you keep up to the different parts of the supposition; yet it is not shewn that this is the only case in which that equality can happen: and indeed the changing a single circumstance will not prevent their equality for you may suppose AB to be unequal to DE and nevertheless the bases may be equal. So that however ready a superficial reader may be to adopt such conclusions, yet the geometrician will find that a great many steps are to be taken before he can demonstrate this to be true. Which is indeed done in the twenty fourth proposition, but after many necessary properties have been previously delivered. If the two sides are equal but the bases unequal it appears from the twenty fifth proposition, that the angle opposite to the greater base is the greater.

But farther, with respect to the angle: I have mentioned already, that we cannot compare angles unless we consider them as belonging to triangles; the sixteenth proposition is made use of to prove that one angle is greater than another: which is a very fundamental proposition and ought to be carefully examined: the simplest view which it is possible to take of it seems to me to be this; to consider the proposition as exprest thus; the angle ACD is greater than CAB and also the angle BCG is greater than ABC; which when properly considered will appear to be the same thing: and then, when we recollect that the angle ACD is equal to BCG, it may certainly be said that ACD is greater than either CAB or ABC: but this is a proposition which ought to be viewed in every light. By this proposition we likewise prove that lines in the situation, which the supposition requires in the twenty seventh can ~~neve~~ form a triangle, and thus leads us to parallelograms; from the never consideration of which we have a new and wonderful property of triangular spaces, namely, that if the base of the triangle continue the same or equal, you may vary the sides until they become a thousand times longer than in their first situation, and still the tri-

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angular ſpace will be the ſame. And this is alſo the principle by which we can change ſpaces from one ſhape to another as will be obvious to every one who underſtands the forty fifth propoſition.

I ſhall conclude this diſſertation, with a general admonition to the reader, to obſerve particularly the uſe of the ſuppoſition in every propoſition; becauſe I know nothing that is ſo eaſily neglected: the conſtruction and even the references to other propoſitions may make a forcible impreſſion upon the mind; but the ſuppoſition has hardly any thing which can awaken our attention to it, except a want of evidence in the concluſion, and there are thoſe who have a wonderful inclination to lay aſide their ſcruples at that part of a demonſtration.

DISSERTATION IV.

WHOEVER has read the first book with care, will perceive the particular use of the triangle for discovering and connecting the different properties of magnitudes; and must at the same time be sensible of the uncommon genius of the author, who could apply this figure in such a manner as to make it discover its own properties, as well as those of other magnitudes. In his second book he presents us with a new instrument, the use of which is no less extensive than that of the triangle; and it is the purpose of this dissertation to draw the attention of the superficial reader to the properties and most obvious uses of the rectangle.

CHAP. I.

Of parallelograms.

TO understand this second book properly, it will be necessary to consider the thirty fifth proposition of the first book, in more points of view than are immediately presented to us by the proposition itself. It is there demonstrated that when the parallelograms are upon the same base, and between the same parallels, the parallelogram spaces are always equal: now this reduces an infinite variety of parallelograms to one single parallelogram as far as the space is concerned; and if a choise of a parallelogram is to be made, which is to represent all the others, the rectangle, or right-angled parallelogram ought to have the preference; because its bounding lines are the least, and its angle fixt or determinate; that its sides are the least will be obvious, by comparing a side of

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the rectangle with the ſide of any other of this infinite variety of parallelograms juſt mentioned; as for inſtance AB with BE, the one of which ſubtends an acute angle and the other a right angle in the ſame triangle.

But this rectangle itſelf, is ſufficiently or rather abſolutely determined by any two of the lines about any one of its angles: and thus two ſtraight lines, will fix an infinite variety of parallelogram ſpaces.

Now this is the firſt definition of the ſecond book, which is thus expreſſed; Every right angled parallelogram, is ſaid to be contained by any two of the ſtraight lines containing the right angle. The ſpace indeed is bounded by four ſtraight lines; but not by four different ſtraight lines, for the oppoſite ſides are the ſame, or equal. Here the reader is carefully to obſerve that there is no difference made here between *equality* and *identity*; for whatever lines are equal, in the preſent caſe they are always conſidered as the ſame; that is if you make a right angled parallelogram, any two ſtraight lines whatever, that are equal to the two lines about one of the angles; are ſaid to contain the ſpace, as well as the particular lines which make two ſides of the very figure.

This circumſtance perplexes beginners exceedingly; and ought to be made the ſubject of frequent meditation before one attempts to read a ſingle propoſition in this book. And here it would be proper to begin with conſidering the nature of a parallelogram, ſtill more particularly: the reader has ſeen what will fix the parallelogram ſpace: he is next to conſider what will fix the whole parallelogram, ſides and angles; and he will firſt obſerve that the two ſides about any of the angles; will determine all the ſides; becauſe the other two ſides are equal to theſe two; next let him try to find out what will determine the angles. Now any angle of a parallelogram being given, it is impoſſible to change any of the others; but this the reader will do well to demonſtrate; which he may do as follows; any two angles of a parallelogram which follow each other in order are equal to two right angles, one of theſe being fixed ſurely the other is fixed alſo; and the oppoſite angles of parallelograms are equal; therefore it is obvious that one angle of the parallelogram fixes all the others. The reader ſhould de-

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monstrate this formally and then he will see that the parallelogram is entirely fixed when an angle, and the two sides about that angle are given. But as I know from experience that the student will not perceive this immediately; let him take only the two sides, and the angle between them; and with these compleat a variety of parallelograms; and he will always find them the same. Not that this construction is necessary towards a right understanding of the conclusion; but I know that it is necessary for fixing the reader's attention; and without attention it is impossible to comprehend any conclusion. Now this point being once settled, it is very clear, that if the parallelogram be right angled, the two sides about one of the angles, or even any lines, equal to these, will fix the parallelegram; from which the full import of this definition may be understood.

CHAP. II.

Containing some remarks on the principles made use of in demonstrating the first eight propositions.

THE simplicity of the demonstrations made use of in the first eight propositions, will be readily acknowledged by every one who understands them: and yet it often happens that the student is at a loss to comprehend their meaning, especially upon the first reading; which is owing to two circumstances; first because he does not understand the first definition; and secondly because we have no common notions of the properties of figures contained in this book; if some property of a triangle should be accidentally mentioned to a person ignorant of the principles of this science, he would nevertheless form some opinion concerning its truth or falshood from some common notions of his own; but read the enunciation of the third proposition in this book to him; and he will have no more opinion of its truth or falshood than if it were delivered to him in an unknown tongue. And supposing these two difficulties to be got over; that is supposing the first definition to be well understood, and the meaning of the proposition become familiar, it is impossible for any evidence to present itself to the

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mind in a simpler form, than that which these demonstrations offer; as the progress of it in general is thus; you describe a certain rectangle or square, which is divided into right angled parallelograms or squares by lines drawn parallel to the sides; and the whole mystery consists in settling what these spaces are according to the first definition. For instance, in the first proposition, as the whole is equal to its parts, the rectangle, or right angled parallelogram, BH is equal to the rectangles BK, DL, EH taken all together; so that we have only to settle what the rectangles BH, BK, DL, EH are according to the first definition, and the demonstration is compleat. In the same manner in the second proposition (which by the bye is only a particular case of the first,) the square ABED is made up of the two rectangles AF and CE; so that we have only to determine what these are according to the first definition, and then the thing proposed is demonstrated. Again in the third proposition, the rectangle AE is made up of the rectangle AD and the square CE; so that we have only to determine what these are according to the same first definition, and the proposition is demonstrated and so in other instances.

The first eight propositions seem to a superficial reasoner to rest in some measure upon a kind of intuitive evidence; because the form of demonstration differs remarkably from that made use of in the remaining propositions of this book; in which the conclusions are inferred from the forty seventh proposition of the first book without exhibiting the squares. But Euclid never draws any consequence from what we are supposed to see, though we ourselves may: for his demonstrations are equally conclusive whether the figures upon which we reason be the objects of our senses or not. The different spaces exhibited in these figures arise from the construction, and have their existence and properties from that, whether we see them or not.

CHAP. III.

Of the addition and substraction of rectangles and squares.

WHEN the learner has made himself master of the demonstrations contained in this second book; it will be proper to turn his thoughts

thoughts to the uſe which may be made of the properties of theſe rectangles and ſquares which he has been conſidering: and as almoſt the ſimpleſt uſe, which can be made of magnitudes, is adding them to one another or taking them from one another; it is very manifeſt that theſe properties will be in a manner uſeleſs, until we have acquired a readineſs in adding them together, and taking them from one another: and this indeed is one of the moſt intricate and extenſive principles in geometry. Much of this buſineſs may, and indeed ought to be learnt from this very book; for whoever ſhould treaſure up in his memory whatever is contained in this book, it would be but uſeleſs lumber unleſs conſidered under this particular point of view. And ſo much for the neceſſity of this practice; but a few examples will beſt explain my meaning.

Suppoſe in the ſecond propoſition you were required to add the rectangle contained by AB and BC to the rectangle contained by AB and AC; theſe taken together make the ſquare of AB: again if from the ſquare of AB you take away the rectangle contained by AB and BC; the remainder is the rectangle contained by AB and AC; all this is obvious from the inſpection of the figure; but as theſe conſequences may be wanted when the figure is not at hand for inſpection, the reader ought to be able to derive them readily from any ſingle ſtraight line cut into two ſegments. But again in the third propoſition if you add the rectangle contained by AC and CB to the ſquare of CB it makes the rectangle contained by AB and BC; or if you add the rectangle contained by AC and CB to the ſquare of AC it makes the rectangle contained by AB and AC: And farther if from the rectangle contained by AB and BC you take away the ſquare of BC there remains the rectangle contained by AC and CB.

In the fourth propoſition if you take the ſquares of AC and CB from the ſquare of AB there is left the rectangle contained by AC and CB taken twice: or if the ſquare of AC be added to the rectangle contained by AC and CB taken twice; their ſum will be equal to the difference between the ſquares of AB and BC; and this is alſo obvious from the inſpection of the figure.

The fifth propoſition, in which is compared the rectangles made by the equal ſegments of a line with thoſe made by the unequal ſegments,

ſegments, furniſhes frequent opportunities for this kind of practice; thus if from the ſquare of half the line BC you take away the ſquare of CD the remainder is the the rectangle contained by AD and DB; and this ſquare of CD added to the rectangle contained by AD and DB makes up the ſquare of half the line. And in the ſixth propoſition if from the ſquare of CD you take the ſquare of CB the remainder is the rectangle contained by AD and DB. One might juſt obſerve upon the fifth; that the rectangle contained by AD and DB and the ſquare of BC are bounded by the ſame extent of line; but that the ſpaces incloſed differ by the ſquare of CD.

It would be tedious to be more particular upon this ſubject; I ſhall therefore conclude with obſerving that ſquares are generally added together or ſubſtracted from one another by the forty ſeventh propoſition of the firſt book; and that they are doubled by the aſſiſtance of a right angled iſoſceles triangle as in the ninth and tenth propoſitions; and may be halved upon the ſame principles.

CHAP. IV.

In which is ſhewn the abſurdity of applying numbers to illuſtrate the propoſitions in this book.

IF you draw two indefinite ſtraight lines at right angles to each other, and cut off from one, ſeven equal parts, beginning at the angular point, and from the other four of the ſame equal parts, and compleat the right angled parallelogram; then, through each of the diviſions, drawing lines parallel to the ſides of the rectangles the whole ſurface is divided into ſquares all equal and twenty eight in number, which is the product of four multiplied by ſeven. And becauſe this parallelogram is equal to any other upon the ſame or an equal baſe and between the ſame parallel lines; and a triangle in the ſame circumſtances, the half of it: therefore this other parallelogram, though not diviſible into ſquares, is ſaid to contain twenty eight ſuch ſquares; and the triangle fourteen of the ſame. And as any rectilineal figure is diviſible into triangles, upon theſe principles

principles any rectilineal figure may have its contents exprest in square measure, inches feet or yards according to the measure by which you suppose your lines to be divided.

Now this way of expressing the surface or contents of a right angled parallelogram in square measure by the product of the two sides about one of the right angles, has introduced a very absurd *practice* upon this second book, which some factious gentlemen have been pleased to stile illustrations.

But we shall be best able to decide how far this method of *operation* can be called an illustration by considering the use for which it is intended. Now the common affairs of life require, that we should pay a particular attention to small and large portions of extension within certain limits; which made it necessary to assume particular parts of extension as a common measure; and for this purpose at first certain parts of the human body seemed to have been used as is obvious from the names of measures; an inch, a foot, a span &c. and very probably he who observed that the feet of different men were unequal in length, and thus proved the necessity of having some fixt standard measure, stood pretty high in the opinion of his cotemporaries for ingenuity. Suppose them to have made the standard measure which we call a foot; for some purposes this would be too large and for others too small. The distance of one place from another exprest in feet could bring no distinct idea to the mind; it would be an improvement to take three feet and make an unit of that, under the different name of a yard, but still more to take one thousand seven hundred and sixty yards and call it a mile. And by such contrivances our perceptions of distance might be made tollerably accurate and suited to our circumstances. Surfaces are measured upon the same plan, and exprest in square feet, yards or miles upon the principles explained above: and to the great comfort of the unthinking part of mankind all surfaces may be compared in a manner sufficiently accurate for the purposes of common life, not only without our ever attending to a surface; but what is more without our having any notion of extension. For our attention is turned to no particular kind of magnitude, as we are only to consider whether one number be greater than another; and here it is not necessary to form your judgement

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by the nature of numbers, but by a particular method of notation which you have been taught.

It would be ſtrange enough to put one upon meaſuring ſuch figures as Euclid treats of in this ſecond book, in order to get at thoſe properties which he ſays belong to them; but it would be the greateſt abſurdity to call this an illuſtration of Euclid's demonſtrations; as the two methods are founded upon principles, in a certain ſenſe directly contrary to each other: theſe menſurations drawing the reader's attention from Euclid's plan to fix it upon ſomething elſe; or, more properly ſpeaking, upon nothing. Neither is the concluſion ſcientifically accurate; for the lines taken at a venture cannot be divided according to any meaſure; particularly if the line be cut as is required in the eleventh propoſition, it is impoſſible to meaſure it and its parts. But to reſt this matter entirely upon the moſt material objection, viz. that it is directly contrary to Euclid's ideas. He teaches how to turn any rectilineal figure into a ſquare in the laſt propoſition of this book. Which ſhews that his plan is to make every ſtep we take the object of the underſtanding, and therefore does not preſent a multitude of things to the mind which it is impoſſible for it to comprehend, but two diſtinct things for its contemplation; he does not expreſs irregular figures by a multitude of ſmall ſquares of any particular name; but reduces any two, which he may have occaſion to compare, to two ſquares; nor has his reaſoning a reference to any particular meaſure. In ſhort if my intention was to purchaſe figures, I might truſt to this practice; but if I meant to reaſon about their properties it muſt be entirely laid aſide.

DISSERTATION V.

THE triangle and rectangle two of the great inſtruments of geometrical inveſtigation, being conſidered, Euclid proceeds to the third, which is of no leſs importance than either of the other two. And is here in its proper place, becauſe the properties of the circle are derived, partly from the triangle, and partly from the rectangle.

In the ſecond book the propoſitions are arranged according to their ſimplicity; and not as in the firſt book according to the dependence which the propoſitions have upon each other: becauſe the tenth might be read firſt, for any connexion which it has with the firſt nine propoſitions; the four laſt indeed are connected with ſome of the preceeding propoſitions, but none of the others. In this third book the arrangement is made partly according to the ſimplicity of the properties and partly according to the connexion which they have with one another; and the explanation of this arrangement with ſome remarks upon the method of demonſtration ſhall be the ſubject of this diſſertation.

CHAP. I.

Containing remarks upon the arrangement of the propoſitions.

IN the two firſt books our author conſiders the circle only as a mechanical inſtrument, and the uſe made of it, reſts entirely upon the third poſtulate; and unleſs it were introduced upon a different footing, it could hardly be reckoned a geometrical figure. In ſhort in the firſt two books it is only a pair of compaſſes with an indefinite extent and perfectly accurate. But here in this book

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it puts on a very different character, and is a geometrical figure bounded by one line called the circumference; and it is alſo added that there is a certain point within the figure from which all the lines drawn to this circumference are equal; but this point is to be found, before we can expect to make much of the properties of the circle. The firſt propoſition teaches us how to find a ſtraight line paſſing through this point; and not ſimply a ſtraight line, but a finite ſtraight line, the cutting which in halves will find the center. This being found it ſeems both natural and neceſſary to conſider whether the circumference of a circle be totally and eſſentially different from a ſtraight line; which it is proved to be in the ſecond, by ſhewing that every ſtraight line joining any two points in the circumference falls entirely within the circle, and can neither apply itſelf to any part of the circumference, nor even meet it except in the two aſſumed points; from whch it follows that a ſtraight line cannot cut the circumference of a circle in more points than two.

But to return to the center; the inveſtigation of this point would ſhew that the center will always be found in any ſtraight line cutting another, terminated by the circle, in halves and at right angles: this diſcovery brings us directly to the third propoſition; for it would not be difficult to imagine that any ſtraight line drawn from the center to the point in which any line was cut in halves, would fall in with this perpendicular, and the contrary; and this would lead to the demonſtration of the fourth propoſition, which muſt have been ſuggeſted already by the property of the center, *that all lines paſſing through it and terminated by the circle are cut in halves there*; becauſe it would be proper to demonſtrate that no other point has this property, and it is here ſhewn that not even two lines terminated by the circle can be cut in halves in any other point. But not to be tedious, the meditating upon the center would very naturally lead to the diſcovery of the firſt fifteen propoſitions; and it is hardly to be imagined that a geometrician would drop the contemplation of the ſecond propoſition before he had diſcovered in what manner a ſtraight line drawn through any point in the circumference of the circle would meet its circumference, which would lead to the conſideration of the ſixteenth, ſeventeenth,

eighteenth

eighteenth and nineteenth propositions. And it would be very proper for the student to examine the first nineteen propositions again and again, before he proceed any farther in the book.

CHAP. II.

The same subject continued.

THE first and second propositions might lead in a very easy and natural manner to the discovery of the first nineteen propositions: but the idea of similar segments is by no means an obvious one; for the common notion is to the last degree vague and undeterminate: so that ever since I could pretend to form any opinion upon this subject, I have always regarded this as one of the most masterly parts of geometry, of the most subtile invention, and as having the most extensive and unexpected consequences; and yet when it is once considered that the angle at the center is double of the angle at the circumference, it all follows in a simple and seemingly obvious manner; for when it is once proved that all angles in the same segment are equal; if they should once begin to conjecture that these angles did not depend upon any particular circle, but were fixt by the segment of the circle which they were considering, they would then arrive at their definition of similar segments, and the whole business was done: and segments of circles could be compared with each other not in a vague manner but accurately and with the same precision as rectilineal figures; and the equality of the circumferences of equal circles could be ascertained in a manner full as satisfactory as the bases of the two triangles are proved to be equal in the fourth proposition of the first book: and thus we could take one circumference double or treble of another, with the same ease that one straight line may be made double or treble of another. It is not easy to say positively, what led them to their definition of similar segments, but the most natural step to it seems to be the proving that the angles in any semicircle are right angles, and that there seemed to be no other circumstance about a segment fixt except the angles which it contained. Those who judge of the demonstration of a proposition by the number

of ſteps which it contains are apt to paſs over the demonſtrations of ſome of theſe propoſitions, as of little difficulty and leſs importance; but to the judicious they will appear, as they really are, of the firſt importance in this ſcience. The learner who would be properly acquainted with this ſubject ſhould conſider the propoſitions from the nineteenth to the thirty fifth, as all tending to the ſame purpoſe, and after giving them a very particular ſeparate examination, carefully trace their connexion with each other.

In the thirty fifth propoſition it is ſaid that the rectangles contained by the ſegments of lines interſecting one another within a circle are always equal; and the thirty ſixth is, that the ſquare of any tangent is equal to the rectangle contained by the whole cutting line, drawn from the ſame point, and the ſegment of it without the circle. From which it follows that if any number of ſtraight lines be drawn from a point without a circle, cutting the circle, the rectangles contained by the whole line and the part without the circle are all equal, which is the ſame property as that delivered in the thirty fifth, when the lines interſect one another within the circle.

To conclude, the reader will perceive upon a careful examination of this book, that there are three great ſources from which the properties of the circle are derived, namely the center; the ſimilar ſegments; and the equal rectangles made by the ſegments of lines terminated by it. We may ſay indeed that they are all ultimately derived from the center; but it appears to me that the ſtudent will find it very uſeful to conſider them under theſe three diſtinct heads.

The plan of the fourth book is ſo very obvious and regular that it ſeems ſufficient juſt to advertiſe the reader, that it is taken up in performing four problems, inſcribing the figure within the circle deſcribing the figure about the circle, inſcribing a circle within the figure and deſcribing a circle about the figure.

CHAP. III.

Containing ſome remarks on the demonſtrations.

EUCLID has a certain idea of demonſtration which he invariably adheres to; and although it be perfectly regular and accurate, yet

yet it does not always take in every circumſtance which beginners require to be put in mind of: this I mention to avoid the imputation of vanity, leſt any additional hints to fit the demonſtration more to the diſpoſition of beginners might be looked upon as an attempt to ſupply ſome defect in the demonſtration: whereas they are intended only to remind the reader, that he may miſtake the meaning of the demonſtration, by truſting to the firſt view which is preſented to his imagination. And I here again repeat, what has been mentioned before, that he ſhould examine the ſimpleſt inſtances firſt, making them gradually more general until he arrive at ſuch a concluſion as the nature of the thing requires, obſerving what conſequences have been drawn from every part of the ſuppoſition, and whether any conſequence depends upon a particular poſition of the lines; and this is to be done not from any hopes of correcting the author, but only to get the better of his own prejudices; becauſe the author's taſk would be endleſs and indefinite, if he ſhould undertake to give ſuch demonſtrations as would remove all the prejudices which we are apt to take up upon this ſubject.

In the ſecond book, there is ſomething ſo determinate in the poſition of the lines, that any variation which the figures can undergo, either in ſhape or ſize, will hardly occaſion any miſconception in the meaning of the propoſition. But this is ſo far from being the caſe with regard to the *figure* which is the ſubject of this book, that by enlarging its ſize it may ſeem to have very different properties: for a circle deſcribed only with *a radius* equal to the ſemidiameter of the Earth, will differ ſo little from a ſtraight line in its circumference, according to vulgar comprehenſion, that perhaps for an hundred yards it would make a more accurate ſtraight line than was ever drawn by any inſtruments: and how are the contacts, interſections and other properties of ſuch circles to be determined but by reaſoning ſcientifically from the nature of the circle. But neither the neceſſity nor beauty of ſuch demonſtrations can be perceived by thoſe who confine their notions to ſuch circles as they can deſcribe with a pair of compaſſes.

I have indeed humoured this kind of prejudice in the deſcription of the figures for the ſecond and ſixteenth propoſitions, becauſe I never found a beginner who could enlarge his notions of theſe figures to make the poſition which the demonſtration requires ſeem probable.

probable. The reader may therefore give these demonstrations a turn more suitable to the figures by considering that when he has proved, in the second proposition that DB is longer than DE and that they are both drawn from the center, and as DB reaches only to the circumference, therefore the shorter line DE cannot extend so far. In the same manner, in the sixteenth DA being proved to be longer than DG; and being both drawn from the center, and DA reaching only to the circumference the line DG cannot reach so far; therefore the point G is within the circle, and consequently the line AF cuts the circle. Let no one imagine that I propose this as a demonstration; I have only exprest myself thus in compliance with vulgar prejudice; for the legitimate demonstrations are those given by Euclid.

In the eighth proposition the reader might just observe, that DK may be proved to be shorter than DL; in the same manner that DF in the former part of the proposition is proved to be less than DE; and it might be farther proper to compare this proposition with the seventh; when it will appear that they both contain the same property of the circle, only the point is taken within the circle in the one and without it in the other; also it may be proper to remark that the point D must be in the plane of the circle, otherwise the line DM would not cut the circumference; and the same is to be observed when we are required to draw a tangent from a point without a circle: and that the tangent itself must be in the plane of the circle. But Euclid takes it uniformly for granted that all his lines are in the same plane until he comes to the eleventh book. In the ninth proposition the reader might suppose the point E to be within the angle ADB and observe what difference it will make in the demonstration; he cannot then prove that DC will be greater than DB and DB than DA; but he can prove that DB is greater than DC which is sufficient for his purpose. It is possible to mistake the meaning of the eleventh proposition, as if it confined the contact of the circles to a single point; but that is not its meaning; suppose any point of contact, the straight line joining the centers will pass through the very point which you take; for if the line FG was also to pass through another point of contact, the absurdity would be just as great, that GD should be equal to

GH

GH and at the fame time greater, as that it fhould be lefs than GH and at the fame time greater. There is fomething rather fingular in the eighteenth propofition, for it feems to me to be nothing but the corollary to the fixteenth propofition; becaufe a tangent there, is a ftraight line at right angles upon the extremity of the diameter; according this notion therefore the one proves that AE is perpendicular to AB; and the other that AB is perpendicular to AE which is certainly the fame thing. Unlefs it fhould be urged in favour of Euclid that the corollary does not fay that every tangent is at right angles to the diameter paffing through the point of contact, but only if it be at right angles it will be a tangent; but it appears to me that there is no room for this diftinction in the prefent cafe; for when we fay that AE is perpendicular to AB, it is not the converfe of that propofition to fay that AB is perpendicular to AE. It would be a proper exercife for the ftudent in reading the feventeenth propofition to prove that two tangents might be drawn from any point without a circle, and that they are equal, though this is not according to Euclid's plan.

I come now to the twentieth propofition, which fhould be examined with the greateft care, by defcribing figures for every diftinct cafe, and varying the pofition of the angular point through the whole circumference BADC: after this is done it will be proper for the ftudent to make an obtufe angle at the circumference, which will give him a particular kind of angle at the center, and it might admit of fome difpute whether it is to be reckoned an angle in Euclid's fenfe of the word; however it may be proved to be double of the obtufe angle at the circumference of the circle. As this is a very fundamental propofition, the reader may juft obferve, (what I have recommended to him for a conftant practice) the confequences which are drawn from every part of the fuppofition; the angles are faid to be at the center and circumference; and to ftand upon the fame circumference. Now if either of thefe circumftances be omitted, the triangles will not be ifofceles &c. The next propofition fhould alfo be particularly examined in all its cafes; when the fegment is greater than a femicircle, equal to a femicircle, and lefs than one; indeed if an angle increafing until the lines take the fame direction is to be confidered as equal to two right

right angles, and if what an angle wants of four right angles is to be regarded as an angle, there is but one case of the proposition; however if these do not make an angle, yet by drawing a line through the center they will be angles in Euclid's sense of the term, and the demonstration of the other two cases will be obvious.

I shall conclude my remarks upon this book by desiring the reader to take notice of a theorem which follows from the thirty third proposition; viz. If two triangles have their bases equal, and also the angles under which the bases are extended; and if they be between the same parallel lines; the two triangles will be equal in every respect. This is mentioned to shew that the joining the properties of the circle to those of the triangle will enlarge our notions very much; because we never could have discovered this by the triangle alone.

In the fifth proposition of the fourth book, *Simson* thinks that Euclid should have proved that the perpendiculars to the sides of the triangles will meet; and it seems to me equally necessary to shew that the tangents will meet in the third proposition; indeed if they did not the angle AKB would cease to be an angle, and AKB would be a straight, but it is equal to the outward angle of a triangle therefore &c. Or in both these instances it may be demonstrated as *Clavius* has done by joining AB in the one; and DE in the other; which brings them to the eleventh common notion.

But although I recommend it to every student to examine all possible suppositions, yet for reasons which have been mentioned already such omissions seem to me very consistent with Euclid's plan of demonstration; for he supposes his reader attentive; and therefore gives him only, what he judged to be the necessary assistance.

THE

ELEMENTS

OF

EUCLID.

BOOK I.

DEFINITIONS.

1. A POINT is that of which no part *can be taken*. 2. But A LINE *has* length without breadth. 3. Alſo extremities of a line *are* points. 4. Any *line* which lies evenly between the points in itſelf is A STRAIGHT LINE. 5. And that is A SURFACE which has length and breadth only. 6. Alſo extremities of a ſurface, *are* lines. 7. Any ſurface which lies evenly between the ſtraight lines in itſelf is A PLANE. 8. But A PLANE ANGLE is the inclination of lines to one another; *that is* of two lines in a plane, meeting each other and not lying in a ſtraight line. 9. Alſo it is called A RECTILINEAL ANGLE, when the lines containing the angle are ſtraight. 10. But when a ſtraight line ſtanding upon a ſtraight line makes the adjacent angles equal to one another; each of the equal angles is A RIGHT ANGLE: and the ſtanding ſtraight line is called A PERPENDICULAR *to that* on

Book I. which it ſtands. 11. An OBTUSE ANGLE is that which is greater than a right angle. 12. But AN ACUTE *angle is* that which is leſs than a right angle. 13. That which is an extremity of any thing is *called* A TERM. 14. The *ſpace* bounded by one or more terms is called a FIGURE. 15. A CIRCLE is a plane figure bounded by one line, which is called a CIRCUMFERENCE: upon which all the ſtraight lines falling, from ONE POINT of thoſe lying within the figure, are equal to one another. 16. And the point is called the CENTER of the circle. 17. But a DIAMETER of a circle is any right line drawn through the center, and terminated both ways by the circumference of the circle: which alſo divides the circle into two equal parts. 18. Alſo A SEMI-CIRCLE is the figure bounded by a diameter, and the circumference of the circle cut off by it. 19. A SEGMENT of a circle is *the figure* bounded by any ſtraight line and the circumference of a circle. 20. The *figures* bounded by ſtraight lines are *called* RECTILINEAL FIGURES. 21. *Of theſe,* ſuch as are *bounded* by three, *are called* TRILATERAL. 22. As thoſe by four QUADRILATERAL. 23. But thoſe bounded by more than four ſtraight lines *are called* MULTILATERAL. 24. Again of trilateral figures, AN EQUILATERAL TRIANGLE particularly is that which has three equal ſides. 25. But an ISOSCELES, that which has only the two ſides equal. 26. And a SCALENE, that which has the three ſides unequal. 27. But moreover of trilateral figures, that again is a RIGHT-ANGLED TRIANGLE, which has a right angle. 28. And an OBTUSE ANGLED *triangle,* that which has an obtuſe angle. 29. Alſo that which has three acute angles, an ACUTE-ANGLED *triangle.* 30. But of quadrilateral figures, that is a SQUARE, which is equilateral and rectangular. 31. And an OBLONG, that which though rectangular is not equilateral. 32. And a RHOMBUS, though equilateral, is not rectangular. 33. But that which is neither equilateral, nor rectangular, having only its oppoſite ſides and angles equal to one another, *is* a RHOMBOIDES. 34. And *all other* quadrilateral figures except theſe may be called TRAPEZIUMS. 35. Any ſtraight lines, which are in the ſame plane, and being produced indefinitely towards both ſides meet each other on neither, are PARALLELS.

POS-

POSTULATES.

1. Let it be taken for granted, that a ſtraight line may be drawn from any one point to any other point. 2. And that a finite ſtraight line may be produced in a ſtraight line continually. 3. Alſo that a circle may be deſcribed with any center and *at any* diſtance.

COMMON NOTIONS.

1. Magnitudes, which are equal to the ſame magnitude, are equal to one another. 2. And if equal ones be added to equal ones, the wholes are equal. 3. And if from equals, equals be taken away, the remainders are equal. 4. And if to unequals, equals be added, the wholes are unequal. 5. Alſo if from unequals, equals be taken away, the remainders are unequal. 6. And the doubles of the ſame are equal to one another. 7. Alſo the halves of the ſame are equal to one another. 8. Magnitudes which fit each other exactly are equal to one another. 9. Alſo the whole is greater than its part. 10. And all right angles are equal to one another. 11. And if a ſtraight line meeting two ſtraight lines, make thoſe angles, which are inward and upon the ſame ſide of it, leſs than two right angles, the two ſtraight lines being produced indefinitely will meet each other on the ſide where the angles are leſs than two right angles. 12. Alſo two ſtraight lines do not incloſe a ſpace.

PROPOSITION I.

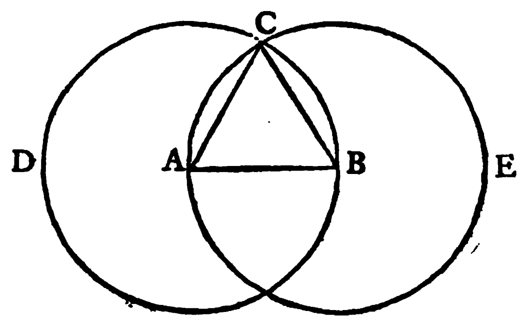
Upon a given finite ſtraight line to deſcribe an equilateral triangle.

Let AB be the given finite ſtraight line: it is required to deſcribe an equilateral triangle upon the ſtraight line AB.

With the center A and at the diſtance AB (by poſt. 3.) let the circle BCD be deſcribed: and again, with the center B, but at

 the

Book I. the diſtance BA (by poſt. 3.) let the circle ACE be deſcribed: and from the point C in which the circles cut one another (by poſt. 1.) let the ſtraight lines CA, CB, be drawn to the points A and B.



Since therefore the point A is the center of the circle BCD (by Def. 15.) AC is equal to AB: again, becauſe the point B is the center of the circle ACE (by Def. 15.) BC is equal to BA. But it has been alſo demonſtrated that CA is equal to AB; each therefore of the *ſtraight lines* CA, CB is equal to AB; but (by com. not. 1.) magnitudes which are equal to the ſame magnitude are equal to one another, and the *ſtraight line* CA is therefore equal to the *ſtraight line* CB; wherefore the three ſtraight lines CA, AB, BC are equal to one another.

The triangle ABC is therefore (by Def. 24.) equilateral, and is deſcribed upon the given finite ſtraight line AB. Which was to be done.

PROP. II.

At a given point, to place a ſtraight line, equal to a given ſtraight line.

Let the given point be the *point* A, and the given ſtraight line the *ſtraight line* BC: it is required at the point A to place a ſtraight line equal to the ſtraight line BC.

For let the ſtraight line AB be drawn (by poſt. 1.) from the point A to the point B: and upon it let the equilateral triangle DAB be deſcribed (by prop. 1.) and let the ſtraight lines AE, BF be produced (by poſt. 2.) in ſtraight lines with DA and DB; and then with the point B as a center and at the diſtance BC, let (by poſt. 3.) the circle CGH be deſcribed: and again, with the point D as a center, and at the diſtance DG let (by poſt. 3.) the circle GKL be deſcribed.

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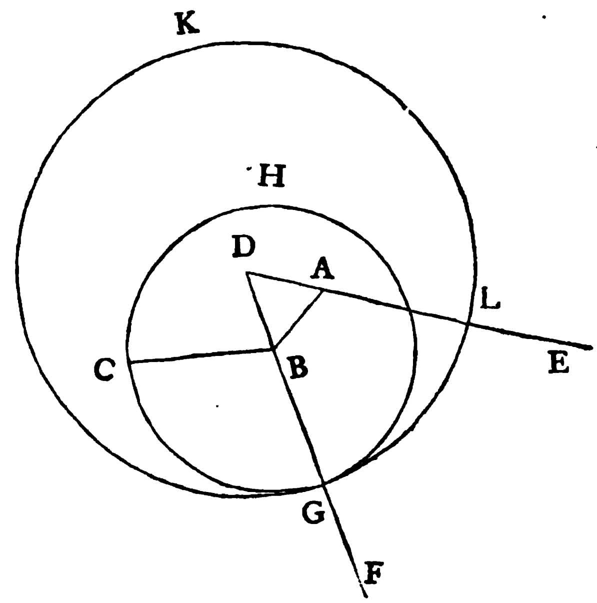


Since therefore the point B is the center of the circle CGH the *ſtraight line* BC is (by def. 15.) equal to BG: and again, becauſe the point D is the center of the circle GKL the *ſtraight line* DL is equal to the *ſtraight line* DG (by Def. 15.) the parts of which viz. the *ſtraight line* DA is equal to the *ſtraigt line* DB, therefore the *ſtraight line* AL the remainder is equal (by com. not. 3.) to the *ſtraight line* BG the remainder, but the *ſtraight line* BC has been ſhewn to be equal to the *ſtraight line* BG: each therefore of the *ſtraight lines* AL, BC is equal to the *ſtraight line* BG; but magnitudes equal to the ſame, are alſo equal to one another: and therefore the *ſtraight line* AL is equal to the *ſtraight line* BC.

Wherefore at the given point A, the ſtraight line AL is placed equal to the given ſtraight line BC. Which was to be done.

PROP. III.

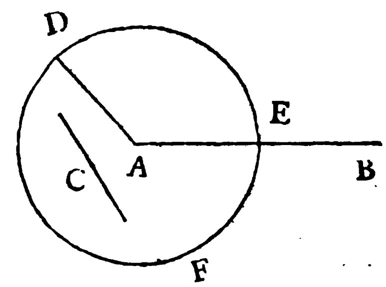
Two unequal ſtraight lines being given, to cut off a part from the greater equal to the leſs.

Let the *ſtraight lines* AB and C be the two unequal ſtraight lines given: it is required from the greater the *ſtraight line* AB to cut off a ſtraight line equal to the ſtraight line C the leſs.

Place (by prop. 2.) at the point A a ſtraight line AD equal to the *ſtraight line* C; and then with the *point* A for a center, but at the diſtance AD, let the circle DEF be deſcribed.

And

Book I. And ſince the point A is the center of the circle DEF, the ſtraight line AE is equal to the ſtraight line AD (by def. 15.) but the *ſtraight line* C is alſo equal to the *ſtraight line* AD: each therefore of the *ſtraight lines* AE and C is equal to the *ſtraight line* AD: wherefore alſo the *ſtraight line* AE is equal (by com. not. 1.) to the *ſtraight line* C.



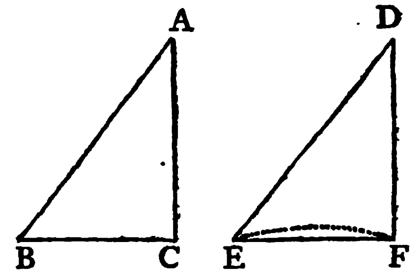
Wherefore two unequal ſtraight lines AB and C being given, from the *ſtraight line* AB the greater, the *ſtraight line* AE has been cut off equal to the *ſtraight line* C the leſs. Which was to be done.

PROP. IV.

If two triangles have the two ſides equal to the two ſides, each to each, and have the angle equal to the angle, the *angle* contained by the equal ſtraight lines: they ſhall alſo have the baſe equal to the baſe, and the triangle ſhall be equal to the triangle; and the other angles under which the *two* equal ſides are extended, ſhall be equal to the other angles, each to each.

Let the *triangles* ABC, DEF be two triangles having the two ſides the *ſtraight lines* AB, AC equal to the two ſides the *ſtraight lines* DE, DF each to each, the one, the *ſtraight line* AB to the *ſtraight line* DE; the other the *ſtraight line* AC to the *ſtraight line* DF; and an angle, that contained by BA, AC *or BAC* equal to that *contained* by ED, DF *or EDF:* I ſay, that alſo the *ſtraight line* BC a baſe is equal to the *ſtraight line* EF a baſe: and the triangle ABC ſhall be equal to the triangle DEF, and the other angles, under which the *two* equal ſides are extended, ſhall be equal to the other angles each to each; the one, the angle *contained* by AB, BC *or ABC* to that *contained* by DE, EF *or DEF*; the other, that *contained* by AC, CB *or ACB* to that *contained* by DF, FE *or* DFE.

For



For the triangle ABC being applied to the triangle DEF; and the point A in particular being put upon the point D, but the ſtraight line AB upon the upon the *ſtraight line* DE, alſo the *point* B will apply itſelf to the *point* E, becauſe the *ſtraight line* AB is equal (by ſuppoſition) to the *ſtraight line* DE: but the *ſtraight line* AB having applied itſelf to the *ſtraight line* DE, the ſtraight line AC will alſo apply itſelf to the *ſtraight line* DF, becauſe the angle *contained* by BA, AC *or BAC* is equal to that *contained* by ED, DF *or EDF* (by ſupp.): wherefore alſo, becauſe again (by ſupp.) the *ſtraight* line AC is equal to the *ſtraight line* DF, the point C will apply itſelf to the point F: certainly alſo the *point* B has applied itſelf to the *point* E, wherefore the *ſtraight line* BC a baſe, will apply itſelf to the *ſtraight line* EF a baſe: for if, the one, the *point* B having applied itſelf to the *point* E, and the other, the *point* C to the *point* F, the *ſtraight line* BC a baſe will not apply itſelf to the *ſtraight line* EF, two ſtraight lines will incloſe a ſpace, which (by com. not. 12.) is impoſſible: therefore the *ſtraight line* BC a baſe will apply itſelf to the *ſtraight line* EF, and (by com. not. 8.) will be equal to it: wherefore alſo the whole, the triangle ABC will apply itſelf to the whole, the triangle DEF, and (by com. not. 8.) will be equal to it: and the other angles will apply themſelves to the other angles, and (by com. not. 8.) will be equal to them; the one *contained* by AB, BC or ABC to that *contained* by DE, EF *or DEF*; the other *contained* by AC, CB *or ACB* to that *contained* by *DF, FE or DFE*.

If therefore two triangles have the two ſides equal to the two ſides, each to each, and have the angle equal to the angle, the *angle* contained by the equal ſtraight lines; they ſhall alſo have the baſe equal to the baſe, and the triangle ſhall be equal to the triangle; and the other angles, under which the *two* equal ſides are extended, ſhall be equal to the other angles, each to each. Which was to be demonſtrated.

PROP.

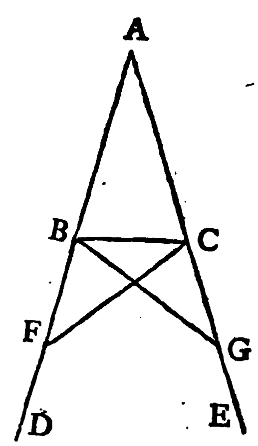
Book I.

PROP. V.

The angles at the baſe of iſoſceles triangles are equal to one another; and if the equal ſtraight lines be produced, the angles under the baſe ſhall be equal to one another.

Let the *triangle* ABC be an iſoſceles triangle, having the ſide AB equal to the ſide AC, and let the ſtraight lines BD, CE be produced in ſtraight lines to AB, AC; I ſay that the one, the angle *contained* by AB·BC *or ABC* is equal to the *angle contained* by AC, CB *or ACB*; the other, *contained* by CB, BD *or CBD* to the *angle contained* by BC, CE *or BCE*.

Let any point which you may accidentally happen upon, as the *point* F be taken in the line BD, and let the *ſtraight line* AG be cut off (by prop. 3.) from the greater the *ſtraight line* AE, equal to the leſs the ſtraight line AF; and let the ſtraight lines FC, GB be drawn.



Since therefore the one *ſtraight line* AF is equal to the ſtraight line AG and the other AB to the *ſtraight line* AC, certainly the two FA, AC, are equal to the two, GA, AB, each to each, and they contain a common angle, the *angle contained* by FA, AG *or FAG:* therefore (by the 4th prop.) the ſtraight line CF a baſe is equal to the *ſtraight line* GB a baſe, and the triangle AFC will be equal to the triangle AGB, and the other angles will be equal to the other angles, each to each, under which the equal ſides are extended, the one *contained* by AC, CF or ACF to the *angle contained* by AB, BG *or ABG*, and the other *contained* by AF, FC *or AFC* to the *angle contained* by AG, GB *or AGB*.

And ſince a whole the *ſtraight line* AF is equal to a whole the *ſtraight line* AG, *parts* of which the *ſtraight line* AB is equal to the *ſtraight line* AC, wherefore (by com. not. 3.) the *ſtraight line* BF a remainder is equal to the *ſtraight line* CG a remainder: but the *ſtraight line* FC has been proved equal to the *ſtraight line* GB; certainly

tainly the two BF, FC are equal to the two CG, GB, each to each; and an angle the *angle contained* by BF, FC or BFC is equal to an angle, the *angle contained* by CG, GB or CGB, and the *ſtraight line* BC is a common baſe of them. And therefore (by the 4th prop.) the triangle BFC will be equal to the triangle CGB, and the other angles, under which the *two* equal ſides are extended, will be equal to the other angles, each to each; wherefore the one *contained* by FBC is equal to the *angle contained* by GCB; and the other *contained* by BCF to the angle contained by CGB. Becauſe therefore a whole the angle *contained* by ABG has been proved equal to a whole the angle *contained* by ACF, *parts* of which, the *angle contained* by CBG is equal to that *contained* by BCF, therefore the *angle contained* by ABC which remains is equal (by com. not. 3.) to that *contained* by ACB which remains, and they are at the baſe of the triangle ABC; but the *angle contained* by FBC has been alſo proved equal to that *contained* by GCB, and they are under the baſe. Book I.

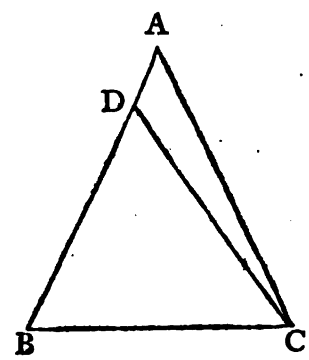
Wherefore the angles at the baſe of iſoſceles triangles are equal to one another; and if the equal ſtraight lines be produced, the angles under the baſe ſhall be equal to one another. Which was to be demonſtrated.

PROP. VI.

If the two angles of a triangle be equal to one another, alſo the ſides extended under the equal angles, ſhall be equal.

Let there be a triangle the *triangle* ABC, having the angle *contained* by ABC equal to the angle contained by ACB: I ſay alſo that a ſide, the *ſtraight line* AC, is equal to a ſide the ſtraight line AB.

For if the *ſtraight line* AC is unequal to the *ſtraight line* AB, the one of them is greater: let the ſtraight line AB be the greater; and (by prop. 3.) let the ſtraight line DB be cut off from the greater the *ſtraight line* AB equal to the leſs the *ſtraight line* AC, and let the *ſtraight line* DC be drawn.



 Since

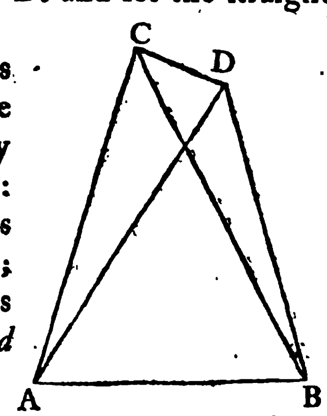
Book I. Since therefore the *ſtraight line* DB is equal to the *ſtraight line* AC, but the *ſtraight line* BC common; certainly the two DB, BC are equal to the two AC, CB, each to each; and an angle, the *angle contained* by DBC is equal to the angle *contained* by ACB (by ſupp.); wherefore (by prop. 4.) the *ſtraight line* DC a baſe is equal to the *ſtraight line* AB a baſe, and the triangle ABC will be equal to the triangle DCB, the greater to the leſs which is abſurd (by com. not. 9.); wherefore the *ſtraight line* AB is not unequal to the *ſtraight line* AC; therefore equal.

If therefore the two angles of a triangle be equal to one another, alſo the ſides, extended under the equal angles, ſhall be equal. Which was to be demonſtrated.

PROP. VII.

Upon the ſame ſtraight line, other two ſtraight lines, equal each to each to the ſame two ſtraight lines, will not join together at different points towards the ſame parts, if they have the ſame extremities as the firſt ſtraight lines.

For if it be poſſible, upon the ſame ſtraight line the *ſtraight line* AB, let other two ſtraight lines, the *ſtraight lines* AD, DB equal, each to each, to the ſame two ſtraight lines, the *ſtraight lines* AC, CB, be joined together at different points, the *points* C and D; the *points* C and D being towards the ſame parts; having the ſame extremities the *points* A, B as the firſt ſtraight lines; ſo that the one CA be equal to the *ſtraight line* DA, having the ſame extremity with it, the *point* A; and the other CB to the *right line* DB, having the ſame extremity with it, the *point* B: and let the ſtraight line CD be drawn.



Since therefore the *ſtraight line* AC is equal to the *ſtraight line* AD, alſo an angle the *angle contained* by ACD is equal (by prop. 5.) to the *angle contained* by ADC: wherefore the *angle contained* by ADC is greater than the *angle contained* by DCB; therefore the *angle contained* by CDB is is greater by much than the *angle contained* by DCB.

Since

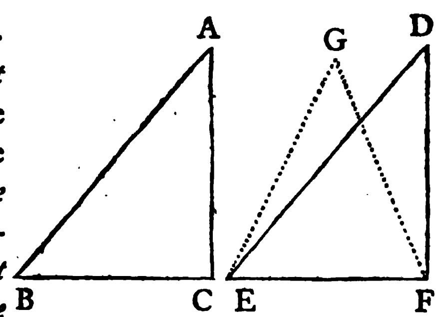
Since again the *straight line* CB is equal to the *straight line* DB (by supp.) also an angle the *angle contained* by CDB is equal (by prop. 5.) to the *angle contained* by DCB: but it has been shewn to be much greater than it, which is impossible. Book I.

Therefore upon the same straight line, other two straight lines equal, each to each, to the same two straight lines will not join together at different points towards the same parts, if they have the same extremities as the first straight lines. Which was to be demonstrated.

PROP. VIII.

If two triangles have *not only* the two sides equal to the two sides, each to each, but have likewise the base equal to the base; they shall also have the angle equal to the angle, the *angle* contained by the equal straight lines.

Let there be two triangles, the *triangles* ABC, DEF having the two sides the *straight lines* AB, AC equal to the two sides the *straight lines* DE, DF, each to each, the one AB to the *straight line* DE, and the other AC to the *straight line* DF; but let them have also a base the *straight line* BC equal to a base the *straight line* EF, I say also that an angle, the *angle contained* by BAC is equal to an angle, the *angle contained* by EDF.



For the triangle ABC being applied to the triangle DEF, and *not only* the point B being put upon the point E, but also the straight line BC being put upon the *straight line* EF, the point C also will apply *itself* to the *point* F, because the *straight line* BC is equal to the *straight line* EF: certainly the *straight line* BC having applied *itself* to the *straight line* EF, also the *straight lines* BA, AC will apply *themselves* to the *straight lines* ED, DF: for if a base as the *straight line* BC should apply *itself* to a base the *straight line* EF, but the sides BA, AC do not apply themselves to the *straight lines* ED, DF, but change their direction *from them* as the *straight lines* EG, GF; upon the

Book I. the ſame ſtraight line, other two ſtraight lines equal, each to each, to the ſame two ſtraight lines will join together at different points towards the ſame parts, having the ſame extremities; but (by prop. 7.) they do not join together; and the ſides BA, AC, cannot therefore not apply *themſelves* to the *ſtraight lines* ED, DF, the baſe BC applying itſelf to the baſe EF, wherefore they do apply themſelves, ſo that alſo an angle the *angle contained* by BAC will apply *itſelf* to the angle *contained* by EDF, and ſhall be equal to it (by com. not. 8.)

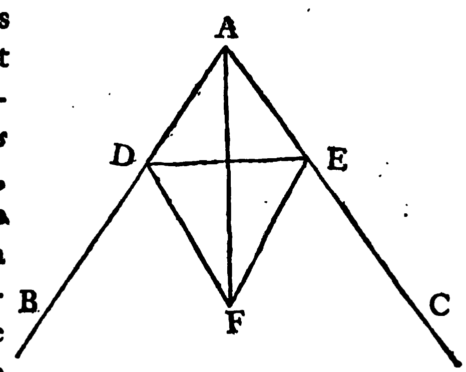
If therefore the two triangles have the two ſides equal to the two ſides, each to each, and have the baſe equal to the baſe; they ſhall alſo have the angle equal to the angle, that which is contained by the equal ſtraight lines. Which was to be demonſtrated.

PROP. IX.

To cut a given rectilineal angle in halves.

Let the given rectilineal angle be, the angle contained by BAC: it is required to cut it in halves.

Let the point D be taken, any accidental *point* in the *ſtraight line* AB; and let the *ſtraight line* AE be cut off (by prop. 3.) from the *ſtraight line* AC equal to the *ſtraight line* AD, and let the ſtraight line DE be drawn (by poſt. 1.) and (by prop. 1.) let the equilateral triangle the *triangle* DEF be deſcribed upon the ſtraight line DE and (by poſt. 1.) draw the ſtraight line AF: I ſay that the angle contained by BAC is cut in halves by the ſtraight line AF.



For ſince the *ſtraight line* AD is equal to the *ſtraight line* AE, but the *ſtraight line* AF common, certainly the two, the *ſtraight lines* DA, AF are equal to the two EA, AF, each to each, and a baſe ~~the~~ the *ſtraight line* DF is equal to a a baſe the *ſtraight line* EF; therefore (by prop. 8.) an angle, the *angle contained* by DAF is equal to an angle, the *angle contained* by EAF.

Wherefore

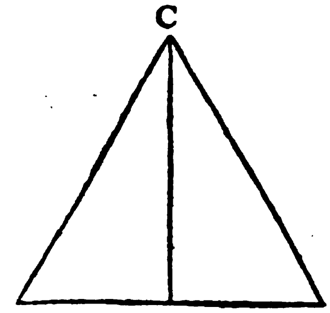
Wherefore the given rectilineal angle, the *angle contained* by BAC is cut in halves by the ſtraight line AF. Which was to be done. Book I.

PROP. X.

To cut a given finite ſtraight line in halves.

Let the *ſtraight line* AB be the given finite ſtraight line: it is required to cut the *ſtraight line* AB in halves.

Let an equilateral triangle the *triangle* ABC be deſcribed (by prop. 1) upon it, and let the angle *contained* by ACB be cut in halves (by prop. 9.) by the ſtraight line CD: I ſay that the ſtraight line AB is cut in halves in the point D.



For ſince the *ſtraight line* AC is equal (by conſtruction) to the *ſtraight line* CB, but the *ſtraight line* CD common, certainly the two AC, CD are equal to the two BC, CD, each to each, and an angle, the *angle contained* by ACD is equal (by conſt.) to the *angle contained* by BCD: therefore (by prop. 4.) a baſe the *ſtraight line* AD is equal to a baſe the *ſtraight line* BD.

Wherefore the given finite ſtraight line the *ſtraight line* AB is cut in halves in the *point* D. Which was to be done.

PROP. XI.

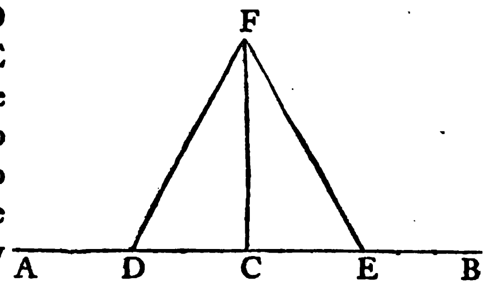
To draw a ſtraight line at right angles to any given ſtraight *line* from any point given in it.

Let the one, the given ſtraight line be the *ſtraight line* AB, and the other the given point in it the *point* C: it is required from the point C to draw a ſtraight line at right angles to the *ſtraight line* AB.

Let any point which you may accidentally happen upon, as the *point* D be taken in the *ſtraight line* AD; and let the ſtraight line CE be placed (by prop. 3.) equal to CD; and let an equilateral triangle the *triangle* DFE be deſcribed upon the *ſtraight line* DE, and let FC be drawn. I ſay that a ſtraight line the *ſtraight line* FC hath

Book I. hath been drawn at right angles to the given ſtraight *line* the *ſtraight line* AB, from any point given in it the *point* C.

For ſince the ſtraight line CD is equal to the *ſtraight line* CE (by conſt.) and the ſtraight line CF common; certainly the two DC, CF are equal to the two EC, CF each to each; and a baſe the *ſtraight line* DF is equal (by conſt.) to a baſe the *ſtraight line* EF: wherefore (by prop. 8.) an angle the *angle contained* by DCF is equal to an angle the *angle contained* by ECF, and they are adjacent or *conſequent:* but when a ſtraight line ſtanding upon a ſtraight line makes the adjacent angles equal to one another; each of the equal angles is a right angle: therefore each of the *angles contained* by DCF, FCE is a right angle.



Wherefore a ſtraight line the *ſtraight line* FC hath been drawn at right angles to the given ſtraight *line* the ſtraight line AB, from any point given in it the *point* C. Which was to be done.

PROP. XII.

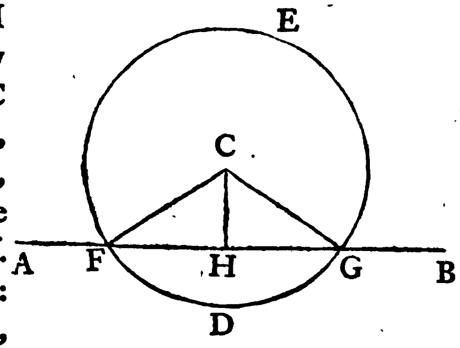
To draw a perpendicular ſtraight line upon any given indefinite ſtraight *line*, from any given point, which is not in it.

Let the one, the given indefinite ſtraight *line*, be the *ſtraight line* AB, and the other, the given point, which is not in it; *be* the *point* C; and it is required upon the given indefinite ſtraight *line*, the *ſtraight line* AB, from the given point, the *point* C, which is not in it, to draw a perpendicular ſtraight line.

For on the other ſide of the ſtraight line AB take any point which you may accidentally happen upon *as* the *point* D and with the *point* C indeed for a center, but at the diſtance CD let a circle be deſcribed, the *circle* FGE; and let the ſtraight line FG be cut in halves (by prop. 10.) in the *point* H, and let the *ſtraight lines* CF, CH, CG be drawn (by poſt. 1.): I ſay that upon the given indefinite ſtraight line, the *ſtraight line* AB, from the given point, the *point* C, which is not in it, a perpendicular hath been drawn, the *ſtraight line* CH.

For

Book I.



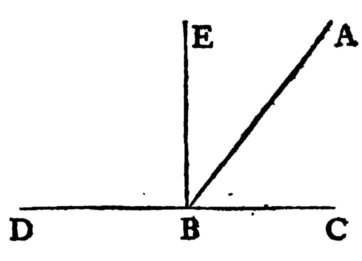
For ſince the ſtraight line FH is equal to the *ſtraight line* HG (by conſt.) and the ſtraight line HC common; certainly the two FH, HC are equal to the two GH, HC, each to each; and a baſe the ſtraight line CF is equal (by def. 15.) to a baſe the *ſtraight line* CG: wherefore (by prop. 8.) an angle, the *angle contained* by CHF is equal to an angle, the *angle contained* by GHC, and they are adjacent: but when a ſtraight line ſtanding upon a ſtraight line makes the adjacent angles equal to one another; each of the equal angles is a right angle: and the ſtanding ſtraight line is called a perpendicular *to that* on which it ſtands.

Therefore upon the given indefinite ſtraight line the *ſtraight line* AB, from the given point, the *point* C, which is not in it, a perpendicular hath been drawn, the *ſtraight line* CH. Which was to be done.

PROP. XIII.

Whenever a ſtraight line ſtanding upon a ſtraight line makes angles: it will make either two right angles, or *angles* equal to two right angles.

For let any ſtraight line, the ſtraight line AB, ſtanding upon a ſtraight line the *ſtraight line* CD, make angles, thoſe *contained* by CBA, ABD: I ſay that the angles *contained* by CBA, ABD are either two right angles, or equal to two right angles.



If indeed the *angle contained* by CBA be equal to the angle contained by ABD, they are (by def. 10.) two right angles: but if not, let the ſtraight line BE be drawn from the point B (by prop. 11.) at right angles to the *ſtraight line* CD: therefore the *angles contained* by CBE, EBD are two right angles: and ſince the *angle contained* by CBE is equal to two, the *angles contained* by CBA, ABE; let the angle *contained* by EBD, a common one be added:

Book I. added: therefore (by com. not. 2.) the *angles contained* by CBE, EBD are equal to three, the *angles contained* by CBA, ABE, EBD. Again, because the *angle contained* by DBA is equal to two, the the angles contained by DBE, EBA; let the angle contained by ABC a common one be added: therefore (by com. not. 2.) the angles, the *angles contained* by DBA, ABC are equal to three, the *angles contained* by DBE, EBA, ABC: but the *angles contained* by CBE, EBD have also been proved equal to the same three: but magnitudes which are equal to the same magnitude, are equal to one another: and the *angles contained* by CBE, EBD are therefore equal to the *angles contained* by DBA, ABC: but the *angles contained* by CBE, EBD are two right angles, and the *angles contained* by DBA, ABC are therefore equal to two right angles.

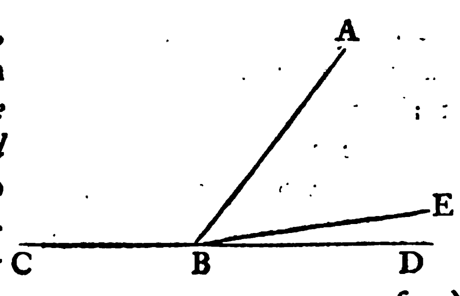
Whenever therefore a straight line standing upon a straight line makes angles; it will make either two right angles, or *angles* equal to two right angles. Which was to be demonstrated.

PROP. XIV.

If, with any straight line, and at the *same* point in it, two straight lines, not lying towards the same parts make the adjacent angles equal to two right angles, the straight lines shall be in a straight line with one another.

For with any straight line, the *straight line* AB, and at the *same* point in it, the *point* B, let two straight lines, the *straight lines* BC, BD not lying towards the same parts, make the adjacent angles, the *angles contained* by ABC, ABD equal to two right angles: I say that the straight line BD is in a straight line with the *straight line* CB.

For if the *straight line* BD is not in a straight line with the *straight line* CB; let the *straight line* BE be in a *straight line* with the straight line CB.



Because therefore a straight line, the *straight line* AB stands upon a straight line, the *straight line* CBE, therefore the angles *contained* by ABC, ABE are equal to two right angles: but the *angles contained* by ABC, ABD are also (by sup.)

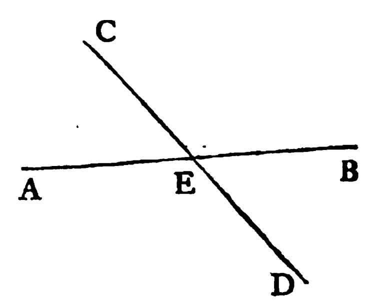
ſup.) equal to two right angles: therefore the *angles contained* by CBA, ABE are (by com. not. 1.) equal to the *angles contained* by CBA, ABD; let *what is* common, the angle *contained* by ABC be taken away, therefore what remains the *angle contained* by ABE is equal (by com. not. 3.) to what remains the angle contained by ABD, the leſs to the greater, which is impoſſible: therefore the ſtraight line BE is not in a ſtraight line with the *ſtraight line* BC: certainly in the ſame manner we ſhall ſhew that neither any other but the *ſtraight line* BD *is:* therefore the *ſtraight line* CB is in a ſtraight line with the *ſtraight line* BD.

Wherefore if, with any ſtraight line, and at the *ſame* point in it, two ſtraight lines, not lying towards the ſame parts, make the adjacent angles equal to two right angles, the ſtraight lines ſhall be in a ſtraight line with one another. Which was to be demonſtrated.

PROP. XV.

If two ſtraight lines cut one another they ſhall make the angles at the vertex equal to each other.

For let two ſtraight lines, the *ſtraight lines* AB, CD cut one another at the point E: I ſay that the one the *angle contained* by AEC is equal to the angle contained by DEB and the other *contained* by CEB to the *angle contained* by AED.



For ſince a ſtraight line, the *ſtraight line* AE ſtands upon a ſtraight line the *ſtraight line* CD, making angles, the *angles contained* by CEA, AED: therefore the angles *contained* by CEA, AED are equal to two right angles (by prop. 13.): again, becauſe a ſtraight line, the *ſtraight line* DE ſtands upon a ſtraight line, the *ſtraight line* AB, making angles the *angles contained* by AED, DEB: therefore the *angles contained* by AED, DEB are equal to two right angles: but alſo the *angles contained* by CEA, AED have been demonſtrated *to be* equal to two right angles: therefore (by com. not. 1.) the *angles contained* by CEA, AED are equal to the *angles contained* by AED, DEB: let the *angle contained* by AED

 which

Book I. *which is* common be taken away, wherefore (by com. not. 3.) the *angle contained* by CEA which remains is equal to the *angle contained* by BED which remains: certainly in the same manner it will be shewn, that also the *angles contained* by CEB, DEA are equal.

If therefore two straight lines cut one another they shall make the angles, at the vertex equal to each other. Which was to be demonstrated.

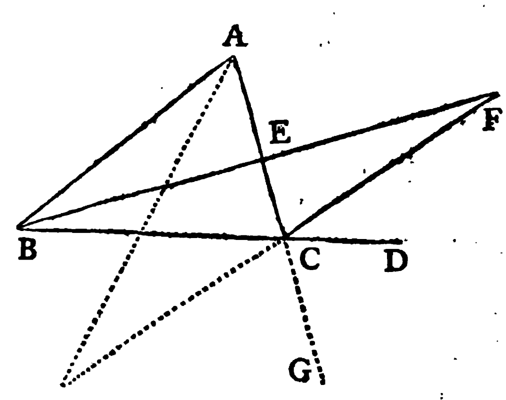
Corollary. Certainly it is manifest from this, that also whatever number of straight lines cut one another, they will make the angles at the section equal to four right angles.

PROP. XVI.

One of the sides of any triangle being produced, the outward angle is greater than either of those *two which are* within and opposite.

Let there be a triangle, the *triangle* ABC, and let one side of it the straight line BC be produced to the *point* D: I say that the angle without, the *angle contained* by ACD is greater than either of those *two which are* within and opposite, the angles contained under CBA, BAC.

Let the straight line AC be cut in halves in the point E (by prop. 10.); and the straight line BE being drawn (by post. 1.) let it (by post. 2.) be produced to the *point* F, and let the straight line EF be placed (by prop. 3.) equal to the *straight line* BE and let the straight line CF be drawn (by post. 1.), and let the straight line AC be produced to G.



Since therefore the one *straight line* AE is equal to the straight line EC (by const.) and the other the *straight line* BE to the *straight line* EF; certainly the two AE, EB are equal to the two CE, EF, each to each, and an angle, the *angle contained* by AEB, is equal to an angle, the *angle*

contained

contained by FEC (by prop. 15.) for *they are* vertical: wherefore (by prop. 4.) a baſe the *ſtraight line* AB is equal to a baſe the ſtraight line CF and the triangle ABE is equal to the triangle FEC, and the remaining angles are equal to the remaining angles, under which the equal ſides are extended, each to each: wherefore the *angle contained* by BAE is equal to the *angle contained* by ECF: but the angle contained by ECD is greater (by com. not. 9.) than the *angle contained* by ECF: wherefore the *angle contained* by ACD is greater than the *angle contained* by BAE: in like manner alſo the ſtraight line BC having been cut in halves, the angle contained by BCG, that is (by prop. 15.) the *angle contained* by ACD, *is* alſo greater than the *angle contained* by ABC. Book I.

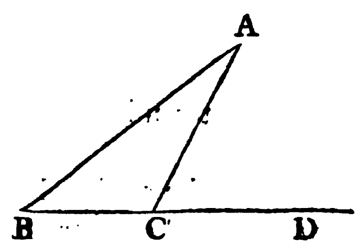
Wherefore one of the ſides of any triangle being produced, the outward angle is greater than either of thoſe *two which are* within and oppoſite. Which was to be demonſtrated.

PROP. XVII.

The two angles of any triangle are leſs than two right angles, being interchanged every way.

Let there be a triangle the *triangle* ABC; I ſay that the two angles of the triangle ABC are leſs than two right angles, being interchanged every way.

For let the *ſtraight line* BC be produced to the *point* D.



And ſince the angle contained by ACD is an outward angle of the triangle ABC, it is greater than the inward and oppoſite, the angle contained by ABC: let a common one be added, the *angle contained* by ACB: therefore the *angles contained* by ACD, ACB are greater (by com. not. 4.) than thoſe *contained* by ABC, BCA; but the *angles contained* by ACD, ACB are equal to two right angles; therefore thoſe *contained* ABC, BCA are leſs than two right angles. And in like manner we ſhall ſhew that the *angles contained* by BAC, ACB are leſs than two right angles, as alſo *thoſe contained* by CAB, ABC.

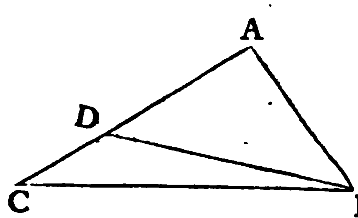
Book I. Wherefore the two angles of any triangle are lefs than two right angles, being interchanged every way.

PROP. XVIII.

The greater fide of every triangle fubtends the greater angle.

Let there be a triangle, the *triangle* ABC, having the fide AC greater than the *fide* AB, I fay that alfo the angle contained by ABC is greater than the angle contained by BCA.

For becaufe the *ftraight line* AC is greater than the *ftraight line* AB let the *ftraight line* AD be placed (by prop. 3,) equal to the ftraight line AB, and let the *ftraight line* BD be drawn.



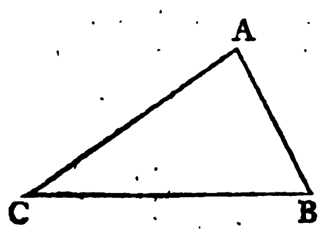
And fince the *angle contained* by ADB is an outward angle of a triangle, the *triangle* BDC, it is greater (by prop. 16.) than the inward and oppofite, the *angle contained* by DCB. But the *angle contained* by ADB is equal to the *angle* contained by ABD (by prop. 5.), becaufe the fide AB is equal to the *fide* AD: therefore alfo the *angle contained* by ABD is greater than the *angle* contained by ACB; wherefore the *angle contained* by ABC is much greater than the *angle contained* by ACB.

Therefore the greater fide, of every triangle, fubtends the greater angle. Which was to be demonftrated.

PROP. XIX.

The greater fide of every triangle is extended under the greater angle.

Let there be a triangle, the *triangle* ABC, having the angle *contained* by ABC greater than the *angle contained* by BCA; I fay that alfo the fide AC is greater than the fide AB.



For if not: certainly the *ftraight line* AC is equal to the *ftraight line* AB, or lefs: but indeed the *ftraight line* AC is not equal to the *ftraight line* AB, for then alfo an angle, the *angle contained* by ABC would be equal (by prop. 5.) to the *angle contained* by ACB: but (by fupp.) it is not: therefore the *ftraight line*

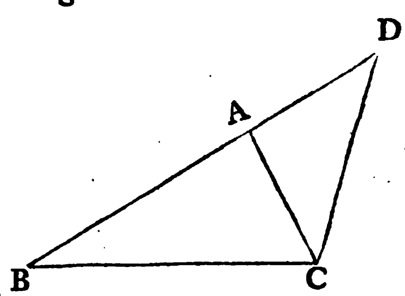
line AC is not equal to the *ſtraight line* AB; neither is the *ſtraight line* AC leſs than the *ſtraight line* AB; for if it were leſs, then (by prop. 18.) an angle the *angle contained* by ABC *would be leſs* than the *angle contained* by ACB: but (by ſupp.) it is not: therefore the *ſtraight line* AC is not leſs than the *ſtraight line* AB; but it has been demonſtrated, that it is not equal *to it*; wherefore the *ſide* AC is greater than the ſide AB. Book I.

Wherefore the greater ſide of every triangle is extended under the greater angle. Which was to be demonſtrated.

PROP. XX.

The two ſides of every triangle are greater than the remaining ſide, being interchanged every way.

For let there be a triangle, the *triangle* ABC: I ſay that the two ſides of the triangle ABC are greater than the remaining *ſide*, being interchanged every way: viz. BA, AC *greater* than BC; and AB, BC *greater* than AC; laſtly BC, CA *greater* than AB.



For let the *ſtraight line* BA be produced to the point D, and let the ſtraight line DA be placed (by prop. 3.) equal to CA, and let the *ſtraight line* DC be drawn.

Since therefore the *ſtraight line* DA is equal to the ſtraight line AC, alſo (by prop. 5.) the *angle contained* by ADC is equal to the *angle contained* by ACD: but the angle *contained* by BCD is greater than the *angle contained* by ACD (by com. not. 9.): wherefore the *angle contained* by BCD is *alſo* greater than the *angle contained* by ADC: and becauſe the *triangle* DCB is a triangle having the angle contained by BCD greater than the *angle contained* by BDC, and (by prop. 19.) the greater ſide is extended under the greater angle, wherefore the *ſtraight line* DB is greater than the *ſtraight line* BC; but the ſtraight line BD is equal to the ſtraight lines AB, AC: therefore the ſtraight lines BA, AC are greater than the ſtraight line BC: certainly in the ſame manner we ſhall demonſtrate that alſo AB, BC are greater than AC; and BC, CA greater than AB. There-

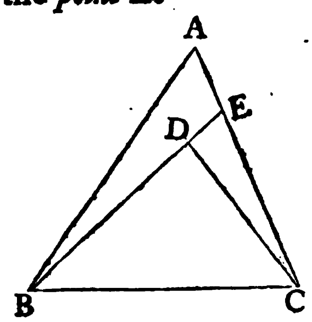
Book I. Therefore of every triangle the two fides are greater than the remaining *fide*, being interchanged every way. Which was to be demonftrated.

PROP. XXI.

If two ftraight lines be joined together within a triangle, upon one of its fides from the extremities of it, the joined lines fhall indeed be lefs than the two remaining fides of the triangle, but will contain a greater angle.

For let two ftraight lines, the *ftraight lines* BD, DC be joined together within a triangle, the *triangle* ABC upon one of the fides the *ftraight line* BC *drawn* from the extremities the *points* B, C; I fay that the *ftraight lines* BD, DC are indeed lefs than the two remaining fides of the triangle the *ftraight lines* BA, AC; but contain a greater angle, the *angle contained* by BDC greater than the *angle contained* by BAC.

For let the *ftraight line* BD be produced to the *point* E.



And becaufe the two fides of any triangle are greater than the remaining *fide*, therefore the two fides of the triangle ABE, the ftraight lines AB, AE are greater than the *ftraight line* BE (by prop. 20.); let the ftraight line EC a common *one* be added: therefore (by com. not. 4.) the *ftraight lines* BA, AC are greater than the *ftraight lines* BE, EC. Again, becaufe the two fides of the triangle CED, the *ftraight lines* CE, ED are greater than the ftraight line CD, let the *ftraight line* DB a common *one* be added; therefore (by com. not. 4.) the *ftraight lines* CE, EB are greater than the *ftraight lines* CD, DB: but the *ftraight lines* BA, AC have been fhewn *to be* greater than the *ftraight lines* BE, EC; therefore BA, AC are greater by much than BD, DC.

Again, becaufe the outward angle of every triangle is greater than the inward and oppofite (by prop. 16.): therefore the outward angle of the triangle CDE, the *angle contained* by BDC is greater than that *contained* by CED: For the fame reafon therefore alfo the outward angle of the triangle ABE, the *angle contained*

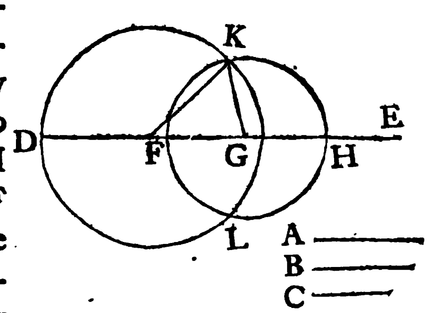
tained by CEB is greater than that *contained* by BAC: but the *angle contained* by BDC has been demonſtrated *to be* greater than that *contained* by CEB: therefore the *angle contained* by BDC is greater by much than that contained by BAC. Book I.

If therefore two ſtraight lines be joined together within a triangle, upon one of its ſides from the extremities of it, the joined lines ſhall indeed be leſs than the two remaining ſides of the triangle, but will contain a greater angle. Which was to be demonſtrated.

PROP. XXII.

To deſcribe a triangle of three ſtraight lines, which are equal to three given ſtraight lines: but it is neceſſary that two of them be greater than the remaining *one*; being interchanged every way.

Let the three given ſtraight lines be A, B, C; of which let the two be greater than the remaining *one*, being interchanged every way, that is A and B greater than C: A and C greater than B; as alſo B and C greater than A: it required to deſcribe a triangle of *ſtraight lines* equal to A, B, C.



Draw any ſtraight line DE terminated at the *point* D but indifinite towards the *point* E; and (by prop. 3.) let DF be made equal to A, and FG equal to B; but GH equal to C: and with the point F for a center, and at the diſtance FD let the circle DKL be deſcribed; and again with the *point* G for a center, and at the diſtance GH let the circle KLH be deſcribed; and draw KF, KG: I ſay that the triangle KFG hath been deſcribed of three ſtraight lines equal to A, B, C.

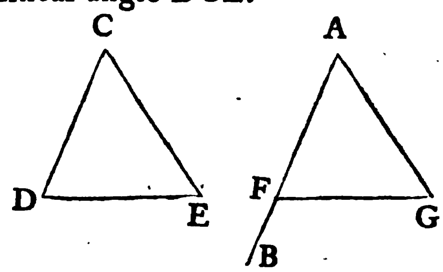
For becauſe the point F is the center of the circle DKL, FD is equal to FK: but FD is equal (by conſt.) to A, therefore FK is equal to A. Again becauſe the point G is the center of the circle LKH, (by def. 15.) GK is equal to GH; but GH is equal to C, therefore GK is equal to C: but FG is alſo equal to B; therefore the three ſtraight lines KF, FG, GK are equal to the three A, B, C. There-

Book I. Therefore the triangle KFC is defcribed of three ftraight lines, which are equal to the three given ftraight lines A, B, C. Which was to be done.

PROP. XXIII.

With a given ftraight line, and at a point in it, to make a rectilineal angle equal to a given rectilineal angle.

Let AB be the given ftraight line, and the *point* A the point in it; and DCE the given rectilineal angle: but it is required with the given ftraight line AB and at the point A in it to make a rectilineal angle equal to the given rectilineal angle DCE.



Take in each of the *lines* CD, CE any points whatever D and E, and join DE, and (by prop. 22.) defcribe the triangle AGF of three ftraight lines, which are equal to the three, CD, DE, CE; fo that CD be equal to AF; CE to AG, and alfo DE to FG.

Since therefore the two DC, CE are equal to the two FA, AG, each to each, and the bafe DE equal to the bafe FG; therefore (by prop. 8.) the angle DCE is equal to the angle FAG.

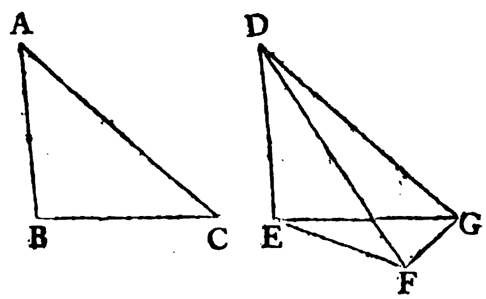
Wherefore with a given ftraight line AB, and at a point in it the *point* A, a rectilineal angle FAG is made equal to the given rectilineal angle DCE. Which was to be done.

PROP. XXIV.

If two triangles have the two fides equal to the two fides, each to each, but have the angle greater than the angle, the *angle* contained by the equal ftraight lines: alfo they will have the bafe greater than the bafe.

Let there be two triangles the *triangles* ABC, DEF having the two fides AB, AC equal to the two fides DE, DF, each to each; AB to DE, and AC to DF; but let an angle the *angle contained* by BAC be greater than the angle *contained* by EDF; I fay that the bafe BC is greater than the bafe EF.

For



For because the angle BAC is greater than the angle EDF; let there be made, with the straight line DE and at the point D in it, the angle EDG equal to the angle BAC; and let DG be made equal (by prop. 3.) to either of the *lines* AC, DF; and let GE, GF be joined.

Since therefore AB is equal to DE and AB to DG; certainly the the two BA, AC are equal to the two ED, DG, each to each; and the angle BAC is equal (by const.) to EDG therefore the base BC is equal to the base EG. Again because DG is equal to DF, the angle DFG is equal (by prop. 5.) to the angle DGF; therefore the *angle* DFG is greater than the *angle* EGF; therefore the *angle* EFG is greater by much than the *angle* EGF; and because there is a triangle, the *triangle* EFG, having the angle EFG greater than the *angle* EGF; but (by prop. 19.) the greater side is extended under the greater angle: therefore the side EG is greater than EF: and EG *is* equal to BC (by part. 1. of this prop.); wherefore also BC is greater than EF.

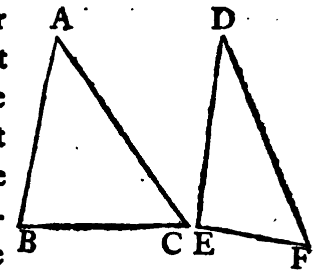
Wherefore if two triangles have the two sides equal to the two sides, each to each, and have the angle greater than the angle, the *angle contained* by the equal straight lines; they will also have the base greater than the base. Which was to be demonstrated.

PROP. XXV.

If two triangles have the two sides equal to the two sides each to each, and have the base greater than the base; they will also have the angle greater than the angle, the *angle* contained by the equal straight lines.

Let there be two triangles the *triangles* ABC, DEF having the two sides BA, AC equal to the two sides DE, DF, each to each, AB to DE and AC to DF; but let the base BC be greater than the base EF: I say also that an angle, the *angle* BAC is greater than an angle, the *angle* EDF.

Book I. For if not; certainly it is equal to it, or lefs. But certainly the angle BAC is not equal to EDF; for then (by prop. 4.) the bafe BC would be equal to the bafe EF: but it is not (by fupp.); therefore the angle BAC is not equal to EDF: but neither indeed is it lefs; for if it were lefs, alfo the bafe BC *would be lefs* (by prop. 24.) than the bafe EF: but it is not (by fupp.); therefore the angle BAC is not lefs than EDF: but it has been demonftrated that neither is it equal: wherefore the angle BAC is greater than EDF.



Wherefore if two triangles have the two fides equal to the two fides each to each, and have the bafe greater than the bafe; they will alfo have the angle greater than the angle, the *angle* contained by the equal ftraight lines. Which was to be demonftrated.

PROP. XXVI.

If two triangles have the two angles equal to the two angles, each to each, and one fide equal to one fide; either the fide which is at the equal angles, or that which is extended under one of the equal angles: they will alfo have the *two* remaining fides equal to the two remaining fides each to each, and the remaining angle *equal* to the remaining angle.

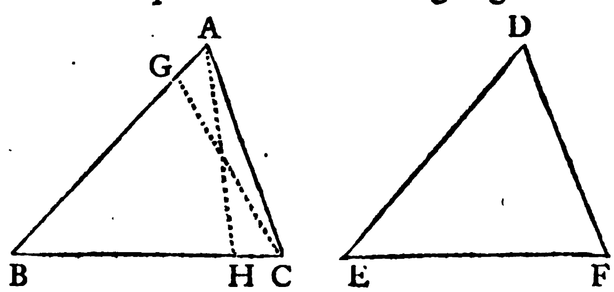
Let there be two triangles the *triangles* ABC, DEF having the two angles ABC, BCA equal to the two *angles* DEF, EFD each to each; the *angle* ABC equal to the *angle* DEF; and the *angle* BCA *equal* to EFD: and let them have alfo one fide equal to one fide:

Firft, the fide at the equal angles, the *ftraight line* BC equal to EF: I fay alfo that they will have the *two* remaining fides equal to the *two* remaining fides, each to each; AB *equal* to DE and AC to DF; and the remaining angle *equal* to the remaining angle, the *angle* BAC *equal* to EDF.

For if AB is unequal to DE, one of them will be greater *than the other*: let AB be the greater and make GB equal (by prop. 3.) to DE, and let CG be joined.

Where-

Wherefore becaufe BG is equal to DE (by conft.) and BC to EF (by fupp.) certainly the two BG, BC are equal to the two DE, EF each to each; and the angle GBC is equal (by fupp.) to DEF: therefore (by prop. 4.) the bafe GC is equal to the bafe DF; and the triangle GCB will be equal to the triangle DEF; and the remaining angles will be equal to the remaining angles, each to each, under which the equal fides are extended: therefore the angle GCB is equal to the *angle* DFE; but the *angle* DFE is fuppofed equal to BCA; therefore BCG is equal to BCA the lefs to the greater, which is impoffible; therefore AB is not unequal to DE; therefore equal: but BC is alfo equal to EF; therefore the two AB, BC are equal to the two DE, EF, each to each, and the angle ABC is equal to the angle DEF; wherefore the bafe AC is equal to the bafe DF and the remaining angle BAC is equal to the remaining angle EDF. Book I.



But again, let the fides extended under the equal angles be equal, as AB *equal* to DE: I fay again that alfo the remaining fides will be equal to the remaining fides; AC to DF and BC to EF; and befides the remaining angle BAC is equal to the remaining angle EDF.

For if BC be unequal to EF, one of them is greater *than the other*; let BC, if it be poffible, be the greater, and make (by prop. 3.) BH equal to EF, and let AH be joined.

And fince BH is equal to EF (by conft.) and AB to DE (by fupp.); certainly the two AB, BH is equal to the two DE, EF, each to each, and they contain equal angles; therefore (by prop. 4.) the bafe AH is equal to the bafe DF, and the triangle ABH is equal to the triangle DEF, and the remaining angles will be equal to the remaining angles, each to each, under which the equal fides are extended; wherefore the angle BHA is equal to EFD; but the *angle* EFD is equal (by fupp.) to the angle BCA; and (by com. not. 1.) BHA is therefore equal to BCA; the outward angle of the triangle AHC viz. BHA equal to the inward and oppofite

 BCA

Book I. BCA, which (by prop. 16.) is impossible: wherefore BC is not unequal to EF, therefore *it is* equal: but AB is also equal to DE; therefore the two AB, BC are equal to the two DE, EF and they contain equal angles; therefore (by prop. 4.) the base AC is equal to the base DF, and the triangle ABC is equal to the triangle DEF and the remaining angle BAC is equal to the remaining angle EDF.

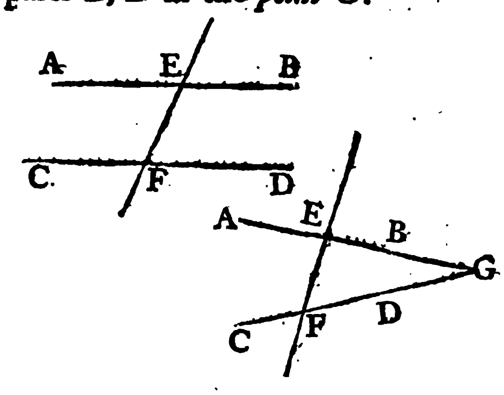
If therefore two triangles have the two angles equal to the two angles, each to each, and have one side equal to one side; either the side which is at the equal angles, or that which is extended under one of the equal angles: they will also have the *two* remaining sides equal to the *two* remaining sides, each to each, and the remaining angle *equal* to the remaining angle. Which was to be demonstrated.

PROP. XXVII.

If a straight line falling upon two straight lines makes the alternate angles equal to one another, the straight lines will be parallel to one another.

For let the straight line EF falling upon the two straight lines AB, CD make the alternate angles AEF, EFD equal to one another. I say that the *straight line* AB is parallel to the straight line CD.

For if not, AB and CD being produced will meet, either towards the the parts B, D or towards A, C: let them be produced and let them meet towards the parts B, D in the *point* G.



Certainly (by prop. 16.) the outward angle AEF of the triangle EFG is greater than the inward and opposite angle EFG; but it is also (by supp.) equal, which is impossible: therefore AB, CD being produced, will not meet towards the parts B, D; certainly in the same manner it will be shewn, that neither *will they meet* towards A, C; but those *straight lines*

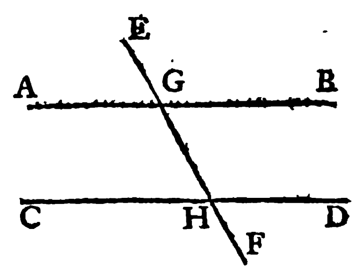
lines meeting one another towards neither parts are parallel: therefore AB is parallel to CD. Book I.

Wherefore if a ſtraight line falling upon two ſtraight lines make the alternate angles equal to one another, the ſtraight lines will be parallel. Which was to be demonſtrated.

PROP. XXVIII.

If a ſtraight line falling upon two ſtraight lines make the outward angle equal to the inward and oppoſite, and towards the ſame parts; or make the inward angles and towards the ſame parts equal to two right angles: the ſtraight lines will be parallel to one another.

For let the ſtraight line EF falling upon the two ſtraight lines AB, CD make the outward angle EGB equal to the inward and oppoſite and towards the ſame parts the angle GHD; or the inward and towards the ſame parts, the *angles* BGH, GHD equal to two right angles; I ſay that AB is parallel to CD.



For becauſe the *angle* EGB is equal to the *angle* GHD; but EGB is equal (by prop. 15.) to AGH, alſo the *angle* AGH is therefore (by com. not. 1.) equal to GHD: and they are alternate; therefore (by prop. 27.) AB is parallel to CD.

Again becauſe the angles BGH, GHD are (by ſupp.) equal to two right angles; and alſo the *angles* AGH, BGH are equal (by prop. 13.) to two right angles; therefore (by com. not. 1.) the *angles* AGH, BGH are equal to the *angles* BGH, GHD: let the common *angle* BGH be taken away, therefore (by com. not. 3.) the remaining *angle* AGH is equal to the remaining *angle* GHD: and they are alternate; wherefore (by prop. 27.) AB is parallel to CD.

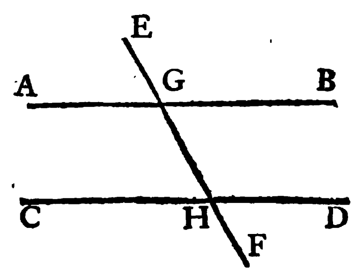
Wherefore if a ſtraight line falling upon two ſtraight lines make the outward angle equal to the inward and oppoſite and towards the ſame parts; or make the inward angles and towards the ſame parts equal

Book I. equal to two right angles: the ſtraight lines will be parallel to one another. Which was to be demonſtrated.

PROP. XXIX.

Or a ſtraight line falling upon the two parallel ſtraight lines makes the alternate angles equal to one another; and the outward *angle* equal to the inward and oppoſite and towards the ſame parts; and the inward angles and towards the ſame parts equal to two right angles.

For let the ſtraight line EF fall upon parallel ſtraight lines, the *ſtraight lines* AB, CD: I ſay that it makes the alternate angles AGH, GHD equal; and the outward angle EGB equal to GHD the inward and oppoſite and towards the ſame parts; and BGH, GHD the inward and towards the ſame parts equal to two right angles.



For if AGH be unequal to GHD, one of them is greater: let AGH be the greater: and becauſe AGH is greater than GHD, let BGH *which is* common be added; therefore (by com. not. 4.) the *angles* AGH, BGH are greater than BGH, GHD: But alſo the *angles* AGH, BGH are (by prop. 13.) equal to two right angles; and therefore the angles BGH, GHD are leſs than two right angles; but thoſe *ſtraight lines* which are produced indefinitely from *angles* leſs than two right angles meet *one another* (by com. not. 11.); therefore AB and CD being produced indefinitely will meet *one another*; but they do not meet, becauſe they are ſuppoſed parallel: therefore AGH is not unequal to GHD, therefore *it is* equal.

But the *angle* AGH is equal to the *angle* EGB (by prop. 15.); and therefore (by com. not. 1.) EGB is equal to GHD.

Let the common *angle* BGH be added: wherefore the *angles* EGB, BGH are equal to the *angles* BGH, GHD: but the *angles* EGB, BGH are (by prop. 13.) equal to two right angles; and therefore the angles BGH, GHD are (by com. not. 1.) equal to two right angles.

Or a ſtraight line, therefore, falling upon the two parallel ſtraight lines, makes the alternate angles equal to one another; and the

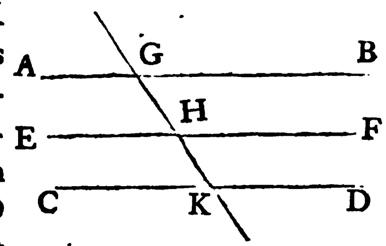
the outward *angle* equal to the inward and oppoſite and towards the ſame parts; and the inward angles and towards the ſame parts equal to two right angles. Which was to be demonſtrated. Book I.

PROP. XXX.

Straight lines parallel to the ſame ſtraight line are alſo parallel to one another.

Let each of the *ſtraight lines* AB, CD be parallel to EF: I ſay alſo that AB is parallel to CD.

For let the ſtraight line GK fall upon them.



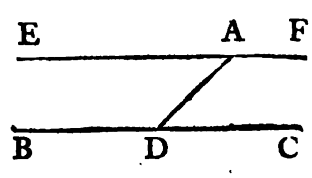
And becauſe the ſtraight line GK hath fallen upon parallel ſtraight lines AB, EF therefore (by prop. 29.) the angle AGH is equal to GHF: again, becauſe the ſtraight line GK hath fallen upon the parallel ſtraight lines EF, CD the angle GHF is equal to GKD: but the angle AGK hath been proved equal to GHF: and therefore (by com. not. 1.) AGK is equal to GKD; and they are alternate *angles:* therefore AB is parallel to CD (by prop. 27.)

Wherefore the *ſtraight lines* which are parallel to the ſame ſtraight line, are parallel to one another. Which was to be demonſtrated.

PROP. XXXI.

Through a given point to draw a ſtraight line parallel to a given ſtraight line

Let the given point be the *point* A; and the given ſtraight line the *ſtraight line* BC; it is required through the point A to draw a ſtraight line parallel to the ſtraight line BC.



Let any point whatever the *point* D be taken in BC; and let AD be joined. And let the *angle* DAE be made, with the ſtraight line AD and at the point A in it, equal to the angle ADC (by prop. 23.): and let the ſtraight line AF be produced in a ſtraight line with AE.

And

Book I. And becaufe the ftraight line AD falling upon the two ftraight lines BC, EF hath made the alternate angles, the *angles* EAD, ADC equal to one another, therefore (by prop. 27.) EF is parallel to BC.

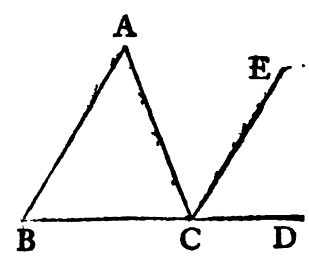
Wherefore through a given point the *point* A a ftraight line EAF hath been drawn parallel to the given ftraight line BC. Which was to be done.

PROP. XXXII.

One of the fides of any triangle being produced, the outward angle is equal to the two inward and oppofite *angles:* and the three inward angles of the triangle are equal to two right angles.

Let there be a triangle, the *triangle* ABC, and let one of its fides BC be produced to the *point* D: I fay that the outward angle ACD is equal to the two inward and oppofite *ones* CAB, ABC; and *that* the three inward angles of the triangle viz. ABC, BCA, CAB are equal to two right angles.

For let CE be drawn (by prop. 31.) through the point C parallel to the ftraight line AB.



And becaufe AB is parallel to CE and AC hath fallen upon them; the alternate angles BAC, ACE are equal (by prop. 27.): Again, becaufe AB is parallel to CE and the ftraight line BD hath fallen upon them; the outward angle ECD is equal to the inward and oppofite ABC (by prop. 29.): but ACE has been alfo fhewn to be equal to BAC; the whole therefore, the outward angle ACD is equal to the two inward and and oppofite ones BAC, ABC.

Let the common angle ACB be added: therefore the angles ACD, ACB are equal to the three angles ABC, BAC, BCA; but ACD, ACB are (by prop. 13.) equal to two right angles; and therefore (by com. not. 1.) ACB, CBA, CAB are equal to two right angles.

Wherefore one of the fides of any triangle being produced, the outward angle is equal to the two inward and oppofite *angles:* and the three inward angles of the triangle are equal to two right angles. Which was to be demonftrated.

PROP.

PROP. XXXIII.

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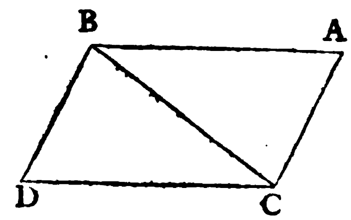
The ſtraight lines joining equal and parallel *ſtraight lines* towards the ſame parts, are alſo themſelves equal and parallel.

Let the *ſtraight lines* AB, CD be equal and parallel; and let the ſtraight lines AC, BD join them towards the ſame parts: I ſay that AC, BD are alſo equal and parallel.



For let BC be joined.

And becauſe AB is parallel to CD, and BC hath fallen upon them; the alternate angles, the *angles* ABC, BCD are (by prop. 29.) equal to one another: and becauſe (by ſupp.) AB is equal to CD, and the *ſtraight line* BC common; certainly the two AB, BC are equal to the two DC, CB; and the angle ABC is equal to BCD; therefore (by prop. 4.) the baſe AC is equal to the baſe BD; and the triangle ABC is equal to the triangle BCD; and the remaining angles will be equal to the remaining angles, under which the equal ſides are extended, each to each; therefore the angle ACB is equal to the *angle* CBD: and becauſe the ſtraight line BC falling upon the two ſtraight lines AC, BD hath made the alternate angles ACB, CBD equal to one another; therefore (by prop. 27.) AC is parallel to BD; and it has been demonſtrated alſo *to be* equal.

Wherefore the ſtraight lines joining equal and parallel *ſtraight lines* towards the ſame parts are alſo themſelves equal and parallel. Which was to be demonſtrated.

PROP. XXXIV.

The oppoſite ſides and alſo the *oppoſite* angles of parallelogram ſpaces are equal to one another, and the diameter cuts the ſpaces in halves.

Let there be a parallelogram, the *parallelogram* ACDB and its diameter the *ſtraight line* BC: I ſay that the oppoſite ſides and alſo the angles of the parallelogram ACDB are equal to one another, and the diameter BC cuts it in halves.

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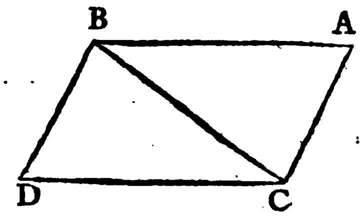


For becauſe AB is parallel to CD and the ſtraight line BC hath fallen upon them; the alternate angles ABC, BCD are equal to one another (by prop. 29.): Again becauſe AC is parallel to BD and BC hath fallen upon them; the alternate angles ACB, CBD are (by prop. 29.) equal to one another: There are two triangles ABC, CBD having the two angles ABC, BCA equal to the two *angles* BCD, CBD, each to each; and one ſide equal to one ſide, the *ſide* at the equal angles, the *ſtraight line* BC common to them both: therefore they will have the remaining ſides equal to the remaining ſides, each to each, (by prop. 26.) and the remaining angle *equal* to the remaining angle; wherefore the ſide AB *is equal* to CD; and AC to BD; and the angle BAC *equal* to the *angle* BDC; and ſince the angle ABC is equal to the *angle* BCD; and the *angle* CBD, to the angle ACB; therefore the whole (by com. not. 2.) angle ACD is equal to the whole ABD; but the angle BAC hath been proved equal to the angle BDC.

Wherefore the oppoſite ſides and alſo the angles of parallelogram ſpaces are equal to one another.

But I ſay alſo that the diameter cuts them in halves: for becauſe AB is equal to CD; and BC common; certainly the two AB, BC are equal to the two DC, CB; each to each; and the angle ABC is equal to the angle BCD; therefore (by prop. 4.) alſo the baſe AC is equal to the baſe BD; and the triangle ABC is equal to the triangle BCD.

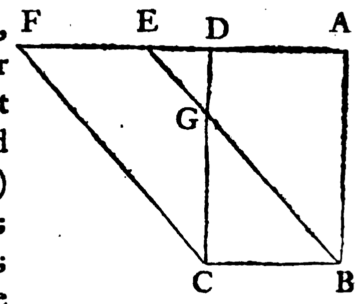
Wherefore the diameter BC cuts in halves the parallelogram ACDB. Which was to be demonſtrated.

PROP. XXXV.

The parallelograms are equal to one another, which are upon the ſame baſe and between the ſame parallel *lines*.

Let there be parallelograms, the *parallelograms* ABCD, EBCF being upon the ſame baſe BC and between the ſame parallel lines AF and BC: I ſay that the *parallelogram* ABCD is equal to the *parallelogram* EBCF.

For

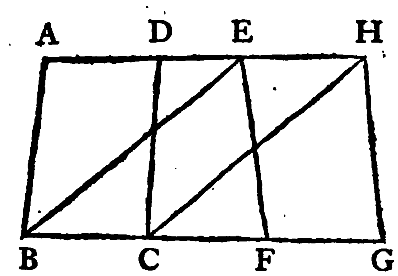


For becaufe ABCD is a parallelogram, AD is (by prop. 34.) equal to BC: and for the fame reafon EF is equal to BC; fo that (by com. not. 1.) AD is equal to EF; and DE is common; therefore (by com. not. 2.) the whole AE is equal to the whole DF; but (by prop. 34.) AB is alfo equal to DC; therefore the two EA, AB are equal to the two FD, DC, each to each; and the angle FDC is equal to the angle EAB (by prop. 29.) the outward to the inward: therefore the bafe EB is equal to the bafe FC; and the triangle EAB is equal to the triangle FCD: let the common *triangle* DEG be taken away; therefore the remainder, the trapezium ABGD is equal to the remainder the trapezium EGCF; let a common part be added, the triangle GBC: therefore (by com. not. 2.) the whole, the parallelogram ABCD is equal to the whole, the parallelogram EBCF. Therefore the parallelograms, being npon the fame bafe, and between the fame parallels are equal to one another. Which was to be demonftrated.

PROP. XXXVI.

The parallelograms, being upon equal bafes and between the fame parallels, are equal to one another.

Let there be parallelograms, the *parallelograms* ABCD, EFGH upon equal bafes the *ftraight lines* BC, FG; and between the fame parallels AH, BG: I fay that the parallelogram ABCD is equal to the *parallelogram* EFGH.



For let BE, CH be joined.

And becaufe BC is (by fupp.) equal to FG, and alfo FG (by prop. 34.) to EH; therefore (by com. not. 1.) BC is equal to EH; but they are likewife (by fup.) parallel; and BE and CH join them; but *ftraight lines* which join equal and parallel ftraight lines towards the fame parts are equal and parallel (by prop. 33.); therefore EB, CH are equal and parallel; wherefore EBCH is a parallelogram, and is equal to

 ABCD

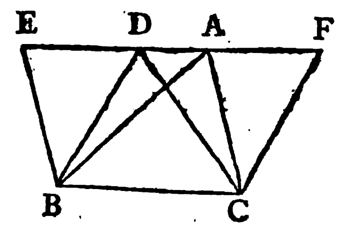
Book I. ABCD (by prop. 35.) for it has the ſame baſe with it the *ſtraight line* BC, and it is between the ſame parallels with it the *ſtraight lines* BC, AH. Certainly for the ſame reaſon alſo EFGH is equal to EBCH; therefore (by com. not. 1.) the parallelogram ABCD is equal to EFGH.

Wherefore the parallelograms, being upon equal baſes and between the ſame parallels are equal to one another. Which was to be demonſtrated.

PROP. XXXVII.

The triangles, which are upon equal baſes and between the ſame parallels, are equal to one another.

Let there be triangles, the *triangles* ABC, DBC, being upon the ſame baſe the *ſtraight line* BC; and between the ſame parallels the *ſtraight lines* AD, BC: I ſay that the triangle ABC is equal to the triangle DBC.



Let AD be produced towards both parts to the points E and F; and through the *point* B let BE be drawn parallel to AC; and through the *point* C let CF be drawn parallel to BD.

Therefore each of the *figures* EBCA, DBCF is (by conſt. and ſup.) a parallelogram: and EBCA is equal (by prop. 35.) to DBCF for they are upon the ſame baſe BC and between the ſame parallels BC, EF: and the triangle ABC is half of the parallelogram EBCA (by prop. 34.), for the diameter AB cuts it in halves; but the triangle DBC is half of the parallelogram DBCF, for the diameter DC cuts it in halves: but the halves of equal magnitudes are equal; wherefore the triangle ABC is equal to the triangle DBC.

Therefore the triangles, which are upon the ſame baſe, and between the ſame parallels, are equal to one another. Which was to be demonſtrated.

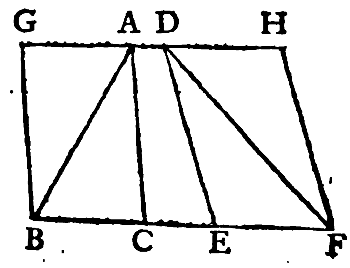
PROP. XXXVIII.

The triangles, which are upon equal baſes and between the ſame parallels, are equal to one another.

Let

Let ABC, DEF be triangles which are upon equal bases BC, EF; and between the same parallels the *straight lines* BF, AD: I say that the triangle ABC is equal to the triangle DEF. Book I.

For let AD be produced towards both parts to G and H; and through B let BG be drawn parallel to CA; and through the *point* F let FH be drawn parallel to DE.



Therefore (by the construction) each of the *figures* GBCA, DEFH is a parallelogram: and (by prop. 36.) GBCA *is* equal to DEFH, for they are upon equal bases BC, EF; and between the same parallels BF, GH: and (by prop. 34.) the triangle ABC is half of the parallelogram GBCA; for the diameter AB cuts it in halves; also the triangle DEF *is* half of the parallelogram DEFH, for the diameter DF cuts it in halves: but (by com. not. 7.) the halves of equal magnitudes are equal: wherefore the triangle ABC is equal to the triangle DEF.

Therefore the triangles, which are upon equal bases and between the same parallels, are equal to one another. Which was to be demonstrated.

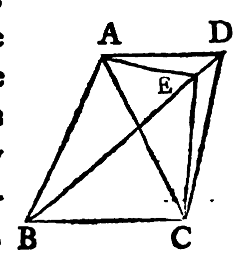
PROP. XXXIX.

The equal triangles, which are upon the same base, and towards the same parts, are between the same parallels.

Let ABC, DBC be the equal triangles, which are upon the same base BC and towards the same parts: I say that they are between the same parallels: for let AD be joined: I say that AD is parallel to BC.

For if *it is* not, let AE be drawn (by prop. 31.) through the point A parallel to the straight line BC, and let EC be joined.

Wherefore (by prop. 37.) the triangle ABC is equal to the triangle EBC, for it is upon the same base with it the *straight line* BC; and between the same parallels BC, AE: but the *triangle* ABC is equal (by sup.) to the *triangle* DBC; therefore (by com. not. 1.) the triangle DBC is equal to the triangle EBC, the greater to the less (by com. not. 9.)



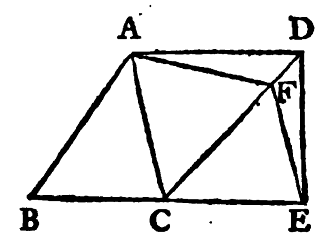
Book I. 9.) which is impoſſible: wherefore AE is not parallel to BC: certainly we ſhall demonſtrate in like manner, that neither is any other *ſtraight line* except AD; therefore AD is parallel to BC.

Wherefore the equal triangles, which are upon the ſame baſe, and towards the ſame parts, are between the ſame parallels. Which was to be demonſtrated.

PROP. XL.

The equal triangles, which are upon equal baſes, and towards the ſame parts, are between the ſame parallels.

Let ABC, CDE be the equal triangles, which are upon equal baſes the *ſtraight lines* BC, CE; and towards the ſame parts: I ſay that they are between the ſame parallels; for let AD be joined I ſay that AD is parallel to BE.



For if *it is* not; let FA be drawn through the *point* A (by prop. 31.) parallel to BE, and let FE be joined.

Therefore (by prop. 38.) the triangle ABC is equal to the triangle FCE; for they are upon the equal baſes BC, CE; and between the ſame parallels BE, AF; but (by the ſup.) the triangle ABC is equal to the triangle DCE; and therefore (by com. not. 1.) the triangle DCE is equal to the triangle FCE; the greater to the leſs (by com. not. 9.) which is impoſſible: wherefore AF is not parallel to BE: certainly we ſhall demonſtrate in the ſame manner that neither *is* any other except the *ſtraight line* AD: therefore AD is parallel to BE.

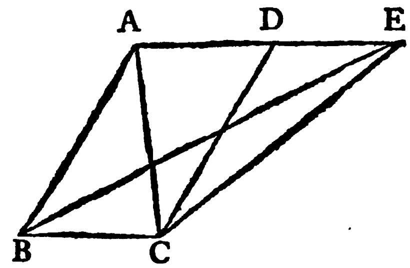
Wherefore the equal triangles, which are upon equal baſes, and towards the ſame parts, are between the ſame parallel lines. Which was to be demonſtrated.

PROP. XLI.

If a parallelogram have the ſame baſe with a triangle; and be between the ſame parallel lines; the parallelogram will be double of the triangle.

For

For let the parallelogram ABCD have the ſame baſe with the triangle EBC, the *ſtraight line* BC; and let them be between the ſame parallels BC, AE: I ſay that the parallelogram ABCD is double of the triangle BEC. Book I.



For let AC be joined.

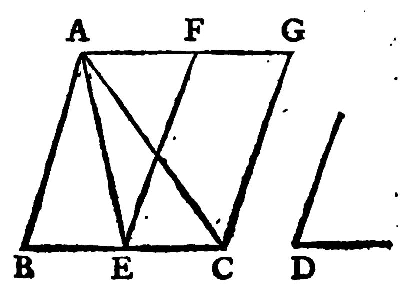
Certainly the triangle ABC is equal to the triangle EBC (by prop. 37.); for it is upon the ſame baſe with it, the *ſtraight line* BC and between the ſame parallels BC, AE; but the parallelogram ABCD (by prop. 34.) is double of the triangle ABC; for the diameter AC cuts it in halves: wherefore the parallelogram ABCD is alſo double of the triangle EBC.

If, therefore, a parallelogram have the ſame baſe with a triangle; and be between the ſame parallels; the parallelogram will be double of the triangle. Which was to be demonſtrated.

PROP. XLII.

To make a parallelogram equal to a given triangle, in a given rectilineal angle.

Let ABC be the given triangle; and D the given rectilineal angle: it is required to make a parallelogram equal to the triangle ABC, in a rectilineal angle equal to D.



Let BC be cut in halves (by prop. 10.) at the *point* E; and let AE be joined; and let the angle CEF be made with the ſtraight line EC and at the point E in it equal to the angle D (by prop. 23.); and through the *point* A let AG be drawn parallel to EC; and through the *point* C (by prop. 31.) let CG be drawn parallel to EF: therefore FECG is a parallelogram.

And becauſe BE is equal to EC, the triangle ABE (by prop. 38.) is alſo equal to the triangle AEC; for they are upon equal baſes BE,

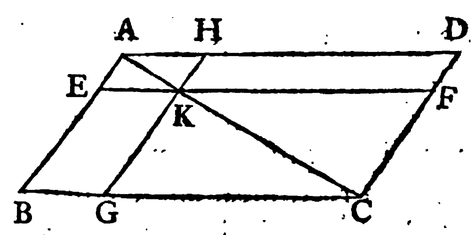
Book I. BE, EC and between the ſame parallel *lines* BC, AG; wherefore the *triangle* ABC is double of the triangle AEC: but the parallelogram FECG is alſo (by prop 41.) double of the triangle AEC: for they have the ſame baſe, and are between the ſame parallels; therefore (by com. not. 6.) the parallelogram FECG is equal to the triangle ABC; and has the angle CEF equal to the given *angle* D.

Therefore a parallelogram the *parallelogram* FECG has been made equal to the given triangle ABC, in an angle the *angle* CEF, which is equal to the *angle* D. Which was to be done.

PROP. XLIII.

The complements of the parallelograms about the diameter of every parallelogram are equal to one another.

Let ABCD be a parallelogram, and AC the diameter of it; and let EH, FG be the parallelograms about AC; and the *parallelograms* BK, KD thoſe which are called complements: I ſay that the complement BK is equal to the complement KD.



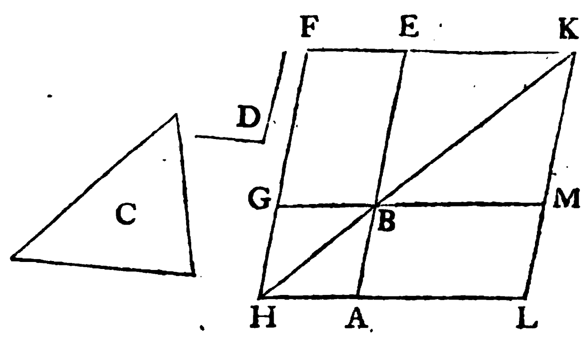
For ſince ABCD is a parallelogram, and AC the diameter of it; the triangle ABC is (by prop. 34.) equal to the triangle ADC: again becauſe EKHA is a parallelogram, and AK the diameter of it; the triangle AEK is (by prop. 34.) equal to the triangle AHK: certainly for the ſame reaſon alſo the triangle KFC is equal to the *triangle* KGC: becauſe therefore the triangle AEK is equal to the triangle AHK, and KFC to KGC; the triangle AEK together with the triangle KGC is equal (by com. not. 2.) to the triangle AHK together with the triangle KFC; but the whole triangle ABC is equal to the whole triangle ADC; therefore (by com. not. 3.) the *remainder*, the complement BK is equal to the remainder the complement KD.

Wherefore the complements of the parallelograms about the diameter of every parallelogram are equal to one another. Which was to be demonſtrated. PROP.

PROP. XLIV.

To apply a parallelogram, to a given ſtraight line, equal to a given triangle, in a given rectilineal angle.

Let the given ſtraight line be AB; but the given triangle the *triangle* C; and the given rectilineal angle, the *angle* D; it is required to apply a parallelogram to the given ſtraight line AB, equal to the given triangle C; in an angle equal to D.



Let the parallelogram BEFG be made (by prop. 42.) equal to the triangle C, in an angle EBG which is equal to D: and let it be placed in ſuch a manner that BE may be in a ſtraight line with AB; and let FG be produced to H; and through the *point* A, let AH be drawn (by prop. 31.) parallel to either of the *ſtraight lines* BG or EF; and let HB be joined.

And becauſe the ſtraight line HF hath fallen upon the parallels AH, EF; therefore the angles (by prop. 29.) AHF, HFE are equal to two right angles: wherefore the angles BHG, GFE are leſs than two right angles: but (by com. not. 11.) thoſe *ſtraight lines*, which are produced indefinitely from angles leſs than two right angles, do meet *each other*; therefore HB, FE being produced will meet: let them be produced, and let them meet in the *point* K; and through the point K, let KL be drawn parallel to either of the *lines* EA, FH; and let GB, HA be produced to the points M, L.

Therefore HLKF is a parallelogram, and HK its diameter; and AG, ME *are* the parallelograms about HK; and LB, BF the *parallelograms* called complements: wherefore (by prop. 43.) LB is equal to BF: but BF is alſo equal to the triangle C; and therefore (by com. not. 1.) LB is equal to C: and ſince the angle GBE is equal (by prop. 15.) to ABM, but GBE is equal (by conſt.) to D; alſo (by com. not. 1.) ABM is equal to the angle D.

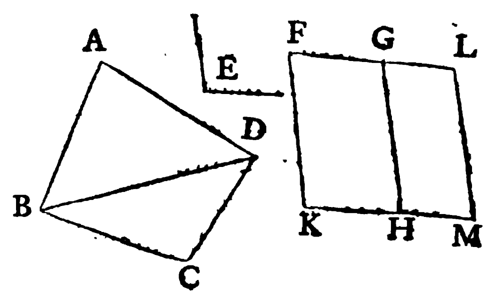
 Where-

Book I. Wherefore a parallelogram LB hath been applied to the given ſtraight line AB, equal to the given triangle C; in an angle ABM, which is equal to the *angle* D. Which was to be done.

PROP. XLV.

To make a parallelogram equal to a given rectilineal *figure* in a given rectilineal angle.

Let the given rectilineal *figure* be ABCD; and E the given rectilineal angle: it is required to make a parallelogram equal to the rectilineal *figure* ABCD in an angle equal to E.



For let DB be joined; and (by prop. 42.) let the parallelogram FH be made equal to the triangle ABD, in the angle HKF, which is equal to E; and (by prop. 44.) let the parallelogram GM, equal to the triangle DBC, be applied to the ſtraight line GH, in the angle GHM, which is equal to E.

And becauſe the angle E is equal to either of the angles HKF, GHM; therefore alſo (by com. not. 1.) HKF is equal to GHM: let the common angle KHG be added; therefore (by com. not. 2.) FKH, KHG are equal to KHG, GHM: but (by prop. 29.) FKH, KHG are equal to two right angles; wherefore alſo KHG, GHM are equal to two right angles: to a certain ſtraight line GH, and to a point in it H, the two ſtraight lines KH, HM not lying towards the ſame parts, make the adjacent angles equal to two right angles; therefore (by prop. 14.) KH is in a ſtraight line with HM: and ſince the ſtraight line GH hath fallen upon the parallels KM, FG; the alternate angles MHG, HGF are equal (by prop. 29.): let the common angle HGL be added: therefore the *angles* MHG, HGL are equal to HGF, HGL; but (by prop. 29.) MHG, HGL are equal to two right angles; therefore alſo HGF, HGL are equal to two right angles; wherefore (by prop. 14.) FG is in a ſtraight line with GL. And becauſe KF is equal and alſo

alſo parallel to GH; but GH *is equal and alſo parallel* to ML; therefore (by com. not. 1. and prop. 30.) KF is equal and parallel to ML; and the ſtraight lines KM, FL join them: and (by prop. 33.) KM, FL are equal and parallel. Wherefore KFLM is a parallelogram; and becauſe the triangle ABD is equal to the parallelogram HF, and the *triangle* DBC to the *parallelogram* GM; therefore the whole rectilineal *figure* ABCD is equal to the whole parallelogram KFLM. Book I.

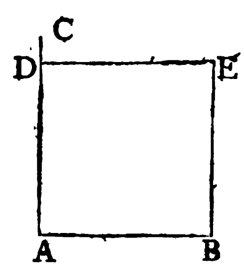
Wherefore a parallelogram KFLM is made equal to a given rectilineal *figure* ABCD in an angle, the *angle* FKM, which is equal to the given *angle* E. Which was to be done.

PROP. XLVI.

To deſcribe a ſquare upon a given ſtraight line.

Let the given ſtraight line be AB: it is required to deſcribe a ſquare upon the ſtraight line AB.

Let AC be drawn (by prop. 11,) at right angles to the ſtraight line AB from the point A in it; and (by prop. 3.) make AD equal to AB: and let DE be drawn, (by prop. 31.) through the point D, parallel to AB; and let BE be drawn through the point B parallel to AD.



Therefore the *figure* ADEB is a parallelogram; wherefore (by prop. 34.) AB is equal to DE; and AD to BE; but alſo AB is equal to AD (by conſt.) therefore the four *ſtraight lines* BA, AD, DE, EB are equal to one another: wherefore the parallelogram ADEB is equilateral; I ſay that it is alſo rectangular: For becauſe the ſtraight line AD has fallen upon the two parallels AB, DE; therefore (by prop. 29.) the angles BAD, ADE are equal to two right angles; but BAD is a right angle (by conſt.); therefore ADE is a right angle; but the oppoſite ſides and alſo the angles of parallelogram ſpaces are equal to one another (by prop. 34.); therefore each of the oppoſite angles is a right angle, the *angles* ABE, BED; wherefore the *figure* ADEB is rectangular; but it has been demonſtrated to be equilateral alſo.

F 2 There-

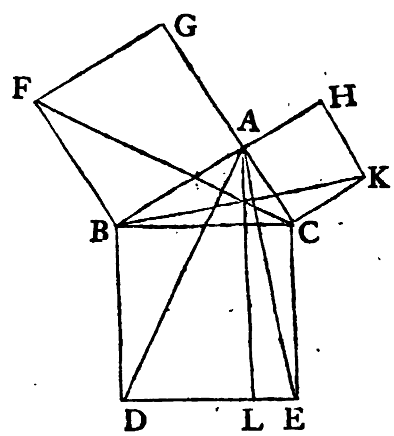
Book I. Therefore (by def. 30.) it is a ſquare; and it hath been deſcribed upon the ſtraight line AB. Which was to be done.

PROP. XLVII.

In right angled triangles, the ſquare of the ſide ſubtending the right angle, is equal to the ſquares of the ſides containing the right angle.

Let ABC be a right angled triangle, having the angle BAC a right *angle:* I ſay that the ſquare of BC is equal to the ſquares of BA, AC.

For (by prop. 46.) let the ſquare BDEC be deſcribed upon BC; and the *ſquares* GB, HC upon BA, AC, and through the *point* A let AL be drawn parallel either to BD or CE; and let AD, FC be joined.



And becauſe each of the angles BAC, BAG is (by conſt. and ſup.) a right *angle*; to a certain *line* BA, and to the point A in it, two ſtraight lines AC, AG, not lying towards the ſame parts, make the adjacent angles equal to two right angles; therefore (by prop. 14.) AC is in a ſtraight line with AG: certainly for the ſame reaſon alſo AB is in a ſtraight line with AH: and ſince the angle DBC is equal to the *angle* FBA (by com. not. 10.), for each *of them* is a right angle, let the common *angle* ABC be added; therefore (by com. not. 2.) the whole *angle* DBA is equal to the whole *angle* FBC; and becauſe the two *ſtraight lines* DB, BA are equal to the two CB, BF, each to each *(being ſides of the ſame ſqu re)* and the angle DBA is equal to the angle FBC; therefore (by prop. 4.) the baſe AD is equal to the baſe FC, and the triangle ABD is equal to the triangle FBC; and the parallelogram BL is double of the triangle ABD (by prop. 41.); for they have the ſame baſe BD and are between the ſame parallels BD, AL: but the ſquare GB is double of the triangle FBC; for again, they have the

the ſame baſe FB, and are between the ſame parallels FB, GC; but the doubles of equal magnitudes are equal to one another (by com. not 6.) wherefore alſo the parallelogram BL is equal to the ſquare GB: Certainly in the ſame manner it will be demonſtrated, having joined AE, BK, that the parallelogram CL is equal to the ſquare HC; therefore the whole ſquare DBCE is equal to the two ſquares GB, HC; and BDEC is the ſquare deſcribed upon BC; and GB, HC the *ſquares deſcribed* upon BA, AC: wherefore the ſquare BE *deſcribed* upon the ſide BC is equal to the ſquares upon the ſides BA, AC. Book I.

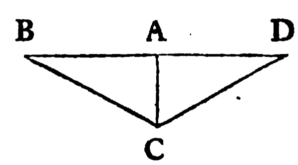
Wherefore in right angled triangles, the ſquare of the ſide ſubtending the right angle is equal to the ſquares of the ſides, containing the right angle. Which was to be demonſtrated.

PROP. XLVIII.

If the ſquare of one of the ſides of a triangle be equal to the ſquares of the two remaining ſides of the triangle; the angle contained by the two remaining ſides of the triangle is a right angle.

For let the ſquare of BC one ſide of the triangle ABC be equal to the ſquares of the ſides BA, AC: I ſay that the angle BAC is a right angle.

For from the point A let the ſtraight line AD be drawn at right angles to AC; and make AD (by prop. 3.) equal to BA; and join DC.



And becauſe DA is equal to AB, the ſquare of AD is equal to the ſquare of AB; let the common ſquare of AC be added; therefore (by com. not. 2.) the ſquares of DA and AC are equal to the ſquares of BA and AC: but the ſquares of DA and AC are (by prop. 47.) equal to the ſquare of DC; for DAC is (by conſt.) a right angle; and the ſquare of BC is equal to the ſquares of BA and AC, for this is ſuppoſed: wherefore (by com. not. 1.) the ſquare of DC is equal to the ſquare of BC; ſo that alſo the ſide DC is equal to BC; and ſince AD is equal to AB and AC common; certainly the two AD, AC are equal

to

Book I. to the two AB, AC, and the base DC is equal to the base BC; therefore the angle DAC is equal to the angle BAC; but DAC is a right angle; therefore the *angle* BAC is a right angle.

Wherefore if the square of one of the sides of a triangle be equal to the squares of the two remaining sides of the triangle; the angle contained by the two remaining sides of the triangle is a right angle. Which was to be demonstrated.

THE

ELEMENTS

OF

EUCLID.

BOOK II.

DEFINITIONS.

1. EVERY right angled parallelogram is ſaid to be contained by the two ſtraight lines containing the right angle. Book II.

2. Of every parallelogram ſpace, let any one of the parallelograms about the diameter of it, together with the two complements, be called a GNOMON.

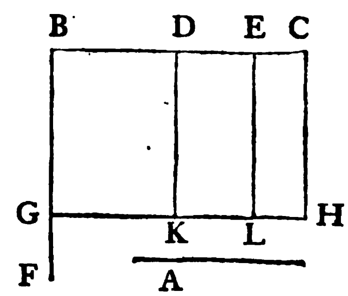
PROP. I.

If there be two ſtraight lines, and one of them be cut into any number of ſegments; the rectangle contained by the two ſtraight lines is equal to the rectangles contained by the undivided line and each of the ſegments *of the divided line*.

Let there be two ſtraight lines A, BC, and let BC be cut as it may happen in the points D and E: I ſay that the rectangle contained

Book II. tained by A and BC is equal to the rectangle contained by A and BD, and to that *contained* by A and DE, and besides to that *contained* by A and EC.

For let BF be drawn from the *point* B at right angles to BC; and let BG be made (by prop. 3. B. 1.) equal to A; and let GH be drawn, through the point G parallel to BC (by prop. 31. B. 1.); and let DK, EL, CH be drawn through the points D, E, C parallel to BG.



Certainly the *rectangle* BH is equal to the rectangles BK, DL, EH; and BH is the rectangle *contained* by A and BC; for (by def. 1. B. 2.) it is contained by GB, BC; but BG is (by conft.) equal to A: and BK is *a rectangle contained* by A and BD; for (by def. 1.) it is contained by GB, BD; and GB is equal to A: but the rectangle DL is *contained* by A and DE; for DK, that is BG (by prop. 34. B. 1.) is equal to A: and also in like manner EH is *a rectangle* contained by A and EC, therefore the rectangle contained by A and BC is equal to the *rectangles contained* by A and BD; and A and DE and also A and EC.

Wherefore if there be two straight lines, and one of them be cut into any number of segments; the rectangle contained by the two straight lines is equal to the rectangles contained by the undivided line and each of the segments *of the divided line*. Which was to be demonstrated.

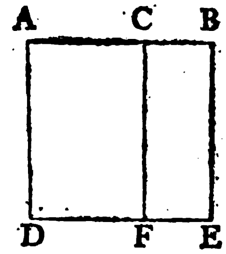
PROP. II.

If a straight line be cut as it may happen, the rectangles contained by the whole and each of the segments are equal to the square of the whole *line*.

For let the straight line AB be cut as it may happen in the point C: I say that the rectangle *contained* by AB and BC; together with the rectangle contained by BA and AC is equal to the square of AB.

For let the square ADEB be described upon AB (by 46. 1.); and let CF be drawn (by 31. 1.) parallel either to AD or BE.

Certainly



Certainly the *rectangle* AE is equal to the *rectangles* AF and CE: and the *rectangle* AE is the square of AB; and the *rectangle* AF is a rectangle contained by BA, AC; for (by def. 1. 2.) it is contained by DA, AC; and AD is equal to AB (being sides of a square): and the rectangle CE is contained by AB, BC; for EB is equal to AB: wherefore the rectangle contained by BA, AC together with that contained by AB, BC is equal to the square of AB.

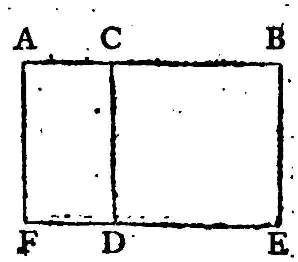
If therefore a straight line be cut as it may happen, the rectangles contained by the whole and each of the segments are equal to the square of the whole *line*.

PROP. III.

If a straight line be cut as it may happen, the rectangle contained by the whole and one of the segments is equal to the rectangle contained by the *two* segments; and to the square of the forementioned segment.

For let the straight line AB be cut as it may happen in the point C: I say that the rectangle contained by AB, BC is equal to the rectangle contained by AC, CB; together with the square of BC.

For let CDEB the square of BC be described (by 46. 1.); and let ED be produced to F: and through the *point* A let AF be drawn (by 31. 1.) parallel to either of the *straight lines* CD, BE.



Certainly the rectangle AE is equal to the *rectangles* AD, CE; and the *rectangle* AE is a rectangle contained by AB, BC; for (by def. 1. B. 2.) it is contained by AB, BE; but BE is equal to BC (being sides of a square): And the rectangle AD is contained by AC, CB; for DC is equal to CB; but the *rectangle* DB is the square of CB: therefore the rectangle contained by AB, CB is equal to the rectangle contained by AC, CB; together with the square of BC.

If therefore a straight line be cut as it may happen, the rectangle, contained by the whole and one of the segments, is equal to the rectangle

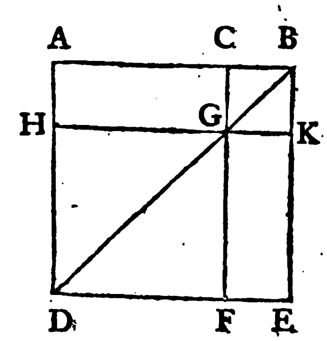
Book II. rectangle contained by the *two* segments; and to the square of the forementioned segment. Which was to be demonstrated.

PROP. IV.

If a straight line be cut as it may happen, the square of the whole *line* is equal to the squares of the *two* segments, and to the rectangle, contained by the *two* segments, *taken* twice.

For let the straight line AB be cut as it may happen in the *point* C: I say that the square of AB is equal to the squares of AC, CB and the rectangle contained by AC, CB *taken* twice.

For let the square ADEB be described upon AB (by 46. 1.); and let BD be joined, and through the *point* C let CGF be drawn parallel to either of the *lines* AD, BE; and through the point G let HK be drawn parallel to either of the *lines* AB, DE.



And because CF is parallel to AD and BD hath fallen upon them; the outward angle BGC is equal to the inward and opposite ADB: but the angle ADB (by 5. 1.) is equal to ABD, because the side BA is equal to AD; therefore (by com. not. 1.) the angle CGB is equal to GBC; so that also (by 6. 1.) the side BC is equal to the side CG: but (by 34. 1.) CG is equal to BK and CB to GK; wherefore also GK is equal to KB: wherefore CGKB is equilateral; I say also that it is rectangular: For because CG is parallel to BK and CB hath fallen upon them; therefore (by 29. 1.) the angles KBC, BCG are equal to two right angles; but KBC is a right angle; wherefore also GCB *is* a right angle; so that also (by 34. 1.) the opposite *angles* CGK, GKB are right angles: wherefore the *figure* CGKB is rectangular; and it has been demonstrated to be equilateral; therefore (by def. 30. 1.) it is a square and it *is described* upon BC: Certainly for the same reason also *the figure* HF is a square; and it is *described* upon HG that is upon AC: therefore the *figures* HF, CK are the squares of AC, CB. And because AG is equal to GE (by 43. 1.); and AG is the *rectangle contained* by AC, CB; for

GC is

GC is equal to CB; and GE is therefore equal to the *rectangle contained* by AC, CB: wherefore AG, GE are equal to the *rectangle contained* by AC, CB *taken* twice; but alſo HF, CK are the ſquares of AC, CB; therefore the four *figures* HF, CK, AG, GE are equal to the ſquares of AC, CB and the *rectangle contained* by AC, CB *taken* twice: but HF, CK, AG, GE are the whole *figure* ADEB, which is the ſquare of AB; wherefore the ſquare of AB is equal to the ſquares of AC, CB and the *rectangle contained* by AC, CB *taken* twice. Book II.

Wherefore if a ſtraight line be cut as it may happen, the ſquare of the whole *line* is equal to the ſquares of the *two* ſegments, and to the *rectangle* contained by the *two* ſegments *taken* twice.

Another Demonſtration.

I ſay that the ſquare of AB is equal to the ſquares of AC, CB and to the rectangle contained by AC, CB *taken* twice.

For, making uſe of the ſame figure, becauſe BA is equal to AD, (by 5. 1.) the angle ABD is equal to the *angle* ADB; and ſince (by 32. 1.) the three angles of every triangle are equal to two right angles; therefore the three angles of the triangle ABD viz. ABD, ADB, BAD are equal to two right angles; but BAD is a right angle (being an angle of a ſquare); therefore the remaining angles ABD, ADB are equal to one right angle; and they are equal: each, therefore, of the angles ABD, ADB is the half of a right angle: but BCG is a right angle (by prop. 29. 1.) for it is equal to the inward and oppoſite *angle* at A; therefore the remainder CGB is the half of a right angle: wherefore the angle CGB is equal to CBG; ſo that alſo (by 6. 1.) the ſide BC is equal to the *ſide* CG; but (by 34. 1.) CB is equal to KG, and CG to BK: wherefore the *figure* CK is equilateral; but it has a right angle, the *angle* CBK: wherefore CK is a ſquare; and it is *deſcribed* upon CB. Certainly for the ſame reaſon alſo HF is a ſquare; and is equal to the ſquare of AC: wherefore CK, HF are ſquares, and are equal to the ſquares of AC, CB: and becauſe AG is equal to GE (by 43. 1.) and AG is the *rectangle contained* by AC, CB; for CG is equal to CB; and EG is equal to the *rectangle contained* by AC, CB; therefore the *figures* AG, GE are equal to the *rectangle contained*

Book II. *tained* by AC, CB; *taken* twice: but CK, HF are equal to the *squares* of AC, CB; therefore the *figures* CK, HF, AG, GE are equal to the *squares* of AC, CB and the *rectangle contained* by AC, CB *taken* twice: But the *squares* CK, HF and the rectangles AG, GE are the whole *square* AE, which is the square of AB.

Wherefore the square of AB is equal to the squares of AC, CB; and to the rectangle contained by AC, CB *taken* twice. Which was to be demonstrated.

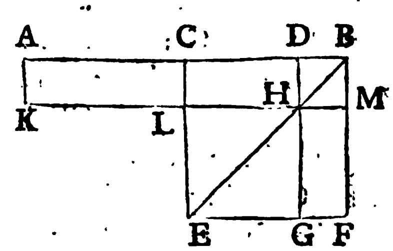
Cor. Certainly from these *demonstrations* it is manifest that in square spaces the parallelograms about the diameter are squares.

PROP. V.

If a straight line be cut into equal and unequal *segments*; the rectangle contained by the unequal segments of the whole *line*, together with the square of the *line* between the *points* of section is equal to the square of half *the line*.

For let any straight line AB be cut into equal *segments* at the *point* C; and into unequal *segments* at the *point* D; I say that the rectangle contained by AD, DB together with the square of CD is equal to the square of CB.

For let the square CEFB be described upon BC; and let BE be joined; and let DHG be drawn, through the *point* D, parallel to either of the *lines* CE, BF; and again through the point H let KLM be drawn parallel to either of the *lines* CB, EF; and again through the *point* A let AK be drawn parallel to either of the *lines* CL, BM.



And since the complement CH is equal to the complement HF; let DM *which is* common be added; therefore the whole CM is equal to the whole DF: but CM is equal to AL (by prop. 36. 1.) because AC is equal to CB: therefore also (by com. not. 1.) AL is equal to DF; let CH *which is* common be added; therefore the whole AH is equal to DF and DL: but AH is the *rectangle contained* by AD, DB for DH is equal to DB (by cor. to 4. 2.): but FD,

FD, DL or FDL is called a Gnomon (by def. 2. 2.); therefore the Gnomon FDL is equal to the *rectangle contained* by AD, DB: Let LG which is common be added; which (by Cor. to 4. 2.) is equal to the ſquare of CD: wherefore the Gnomon FDL and the *ſquare* LG is equal to the rectangle contained by AD, DB and the ſquare of CD: but the Gnomon FDL and the ſquare LG is the whole ſquare CEFB, which is the *ſquare* of CB: Wherefore the rectangle contained by AD, DB together with the ſquare of CD is equal to the ſquare of CB. Book II.

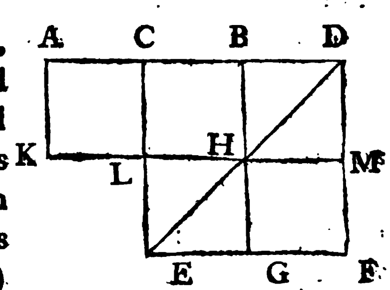
Wherefore if a ſtraight line be cut into equal and unequal *ſegments*; the rectangle contained by the unequal ſegments of the whole *line*, together with the ſquare of the *line* between the *points* of ſection, is equal to the ſquare of half *the line*. Which was to be demonſtrated.

PROP. VI.

If a ſtraight line be cut in halves, and any ſtraight line be added to it in a ſtraight line; the rectangle contained by the whole with the part produced, and the part produced, together with the ſquare of half the *line*; is equal to the ſquare deſcribed upon the compounded *line*, that is of half the line and the part produced, as upon one *line*.

For let any ſtraight line AB be cut in halves at the point C, and let any ſtraight line BD be added to it in a ſtraight line: I ſay that the rectangle contained by AD, DB together with the ſquare of BC is equal to the ſquare of CD.

For let the ſquare CEFD be deſcribed (by 46. 1.) upon CD; and let DE be joined; and through the point B let (by 31. 1.) BHG be drawn parallel to either of the *lines* CE, DF; and through the point H, let KLM be drawn parallel to either of the *lines* AD, EF; and farther through the *point* A let AK be drawn parallel to either of the *lines* CL, DM.



Therefore becauſe AC is equal to CB, the *rectangle* AL (by prop. 36. 1.) is equal to the *rectangle* CH; but CH is alſo equal (by 43. 1.) to HF; wherefore AL is equal to HF; let CM *which is* common be added, therefore the whole AM is equal to CM, MG that is (by def. 2. 2.)

Book II. to the Gnomon CMG: but AM is the *rectangle contained* by AD, DB (by def. 1.2.) for DM is equal to DB (by cor. to 4.2.); therefore the Gnomon CMG is equal to the rectangle contained by AD, DB: let GL *which is* common be added; which is equal (by cor. to 4. 2.) to the ſquare of CB: therefore the rectangle contained by AD, DB together with the ſquare of CB is equal to the Gnomon CMG and the *ſquare* LG: but the Gnomon CMG and the *ſquare* LG is the whole ſquare CEFD, which is the ſquare of CD: wherefore the rectangle contained by AD, DB together with the ſquare of CB is equal to the ſquare of CD.

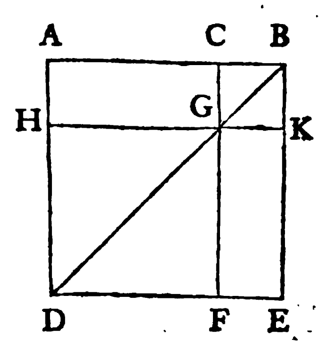
Wherefore if a ſtraight line be cut in halves, and any ſtraight line be added to it in a ſtraight line; the rectangle contained by the whole with the part produced and the part produced; together with the ſquare of half of the *line*; is equal to the ſquare deſcribed upon the compounded *line*; that is of half *the line* and the part produced, as upon one *line*. Which was to be demonſtrated.

PROP. VII.

If a ſtraight line be cut as it may happen; the *ſquare* of the whole *line* and the *ſquare* of one of the ſegments, theſe ſquares, taken together, are equal to the rectangle contained by the whole, and the ſaid ſegments, *taken* twice and the ſquare of the remaining ſegment.

For let any ſtraight line AB be cut as it may happen in the point C; I ſay that the ſquares of AB, BC are equal to the rectangle contained by AB, BC taken twice and the ſquare of AC.

For let a ſquare ADEB be deſcribed upon AB: and let the figure be conſtructed.



And becauſe AG is equal to GE (by 43. 1.); let CK which is common be added: therefore the whole AK is equal to the whole CE; therefore AK, CE are the double of AK: but AK, CE is the Gnomon AKF: and the ſquare CK: therefore the Gnomon AKF and the ſquare CK are double of the rectangle AK: but the double of AK is alſo equal to the rectangle contained by AB,

BC *taken* twice; for BK is (by cor. to 4. 2.) equal to BC: Wherefore the Gnomon AKF and the ſquare CK are equal to the rectangle contained by AB, BC *taken* twice: Let HF which is common be added, which is the ſquare of AC: therefore the Gnomon AKF and the ſquares CK and HF are the whole *ſquare* ADEB and the *ſquare* CK, which are the ſquares of AB, BC: wherefore the ſquares of AB, BC are equal to the rectangle contained by AB, BC *taken* twice together with the ſquare of AC. Book II.

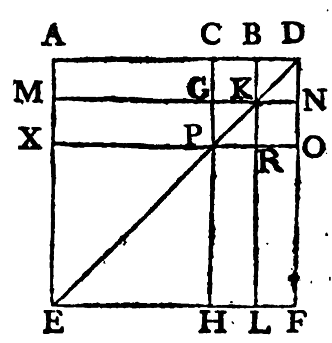
Wherefore if a ſtraight line be cut as it may happen; the *ſquare* of the whole *line* and the ſquare of one of the ſegments, theſe ſquares, taken together, are equal to the rectangle contained by the whole and the ſaid ſegment, taken twice; and the ſquare of the remaining ſegment. Which was to be demonſtrated.

PROP. VIII.

If a ſtraight line be cut as it may happen; the rectangle contained by the whole *line* and one of the ſegments, *taken* four times, together with the ſquare of the remaining ſegment, is equal to the ſquare deſcribed upon the whole and the ſaid ſegment, as upon one *line*.

For let any ſtraight line AB be cut as it may happen in the point C: I ſay that the rectangle contained by AB, BC; *taken* four times, together with the ſquare of AC is equal to the ſquare deſcribed upon AB, BC; as upon one line.

For let BD be produced in a ſtraight line with AB; and make BD equal to CB; and let the ſquare AEFD be deſcribed upon AD: and let the double figure be conſtructed.



Therefore becauſe CB is equal to BD; but (by 34. 1.) CB is equal to GK, and BD to KN; alſo (by com. not. 1.) GK is equal to KN. Certainly for the ſame reaſon PR is equal to RO: and becauſe CB is equal to BD, and GK to KN; therefore alſo the *figure* CK is equal to BN (by 36. 1.), and GR to RN; but the *figure* CK is equal to RN (by 43. 1.); for they are

Book II. are the complements of the parallelogram CO: wherefore BN is alſo equal to GR (by con. not. 1.): therefore the four *figures* the *rectangles* BN, KC, GR, RN are equal to one another; the four therefore are quadruple of CK: Again becauſe CB is equal to BD; but BD (by cor. to 4. 2.) is equal to BK, that is to CG (by 34 1.); and CB to GK, that is, is equal to GP; therefore CG is equal to GP: And becauſe CG is equal GP; and PR to RO; alſo the *rectangle* AG is equal (by 36. 1.) to MP: and PL to RF; but MP and PL are equal (by 43. 1.), for they *are* the complements of the parallelogram ML: therefore alſo AG is equal to RF: wherefore the four *figures*, the rectangles AG, MP, PL, RF are equal to one another: therefore the four *figures* are quadruple of AG: but it has been demonſtrated that the four CK, BN, GR, RN are quadruple of CK; therefore the eight which contain the Gnomon AOH are quadruple of AK; and becauſe AK is the *rectangle contained* by AB, BD; for BK is equal to BD; therefore the rectangle contained by AB, BD *taken* four times is equal to the Gnomon AOH: let XH which is common be added, which is equal to the ſquare of AC: wherefore the rectangle contained by AB, BD *taken* four times, together with the ſquare of AC is equal to the Gnomon AOH and the *ſquare* XH: but the Gnomon AOH and the *ſquare* XH is the whole ſquare AEFD, which is the *ſquare* of AD: therefore the rectangle contained by AB, BC *taken* four times together with the ſquare of AC is equal to the ſquare of AD, that is to the ſquare deſcribed upon AB and BC as upon one *line*.

Wherefore if a ſtraight line be cut as it may happen; the rectangle contained by the whole *line*, and one of the ſegments; taken four times; together with the ſquare of the remaining ſegment, is equal to the ſquare deſcribed upon the whole and the ſaid ſegment, as upon one *line*. Which was to be demonſtrated.

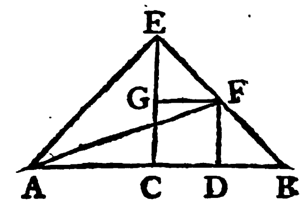
PROP. IX.

If a ſtraight line be cut into equal and unequal *ſegments*; the ſquares, of the unequal ſegments of the whole *line*, is double both of the ſquare of half *the line* and of the *line* between the *points* of ſection.

For

For let the ſtraight line AB be cut into equal ſegments at the *point* C, and into unequal ones at the point D: I ſay that the ſquares of AD, DB are double of the ſquares of AC, CD. Book II.

For from the *point* C, let CE be drawn at right angles to AB; and let it be made equal to either of the lines AC, CB; and let EA, EB be joined; and through the point D let DF be drawn parallel to EC; and through the point F, let FG be drawn parallel to AB, and let AF be joined.



And becauſe AC is equal to CE, the angle EAC (by 5. 1.) is equal to AEC: and ſince the angle at C is a right angle, therefore the remaining angles AEC, EAC are equal to one right angle (by 32. 1.): therefore each of the angles AEC, EAC is the half of a right angle. Certainly for the ſame reaſon alſo each of the *angles* CEB, EBC is the half of a right angle; wherefore the whole angle AEB is a right angle: and becauſe GEF is half a right angle, and EGF a right angle (by. 29. 1.) for it is equal to the inward and oppoſite ECB; therefore the remainder EFG is half a right angle (by 32. 1.); therefore the angle GEF is equal to the *angle* EFG; ſo that alſo (by 6. 1.) the ſide EG is equal to the ſide GF: Again becauſe the angle at B is half a right angle, and FDB a right angle (by 29. 1.) for again it is equal to the inward and oppoſite ECB; therefore the remaining *angle* BFD is half a right angle: therefore the angle at B is equal to the angle DFB; ſo that alſo the ſide DF is equal (by 6. 1.) to the ſide DB: And becauſe AC is equal to CE: the *ſquare* of AC is alſo equal to the *ſquare* of CE; therefore the ſquares of AC, CE; are double of the ſquare of AC: but the ſquare of EA is equal to the *ſquares* of AC, CE (by 47. 1.) for the angle ACE is a right angle: wherefore the ſquare of EA is double of the ſquare of AC: Again becauſe EG is equal to GF, the *ſquare* of EG is equal to the *ſquare* of GF; therefore the ſquares of EG, GF are double of the ſquare of GF; but the ſquare of EF is equal to the ſquares of EG, GF (by 47. 1.); therefore the ſquare of EF is double of the ſquare of GF; but (by 34. 1.) GF is equal to CD; therefore the *ſquare* of EF is double the *ſquare* of CD: but the *ſquare* of AE is alſo the

Book II. double of the *square* of AC; wherefore the squares of AE, EF are double of the squares of AC, CD; but the square of AF is equal (by 47. 1.) to the squares of AE, EF; for the angle AEF is a right angle; therefore the square of AF is double of the *squares* of AC, CD: but the *squares* of AD, DF are equal (by 47. 1.) to the *square* of AF, for the angle at D is a right angle; therefore the *squares* of AD, DF are double of the squares of AC, CD: but DF is equal to DB; wherefore the squares of AD, DB are double of the squares of AC, CD.

Wherefore if a straight line be cut into equal and unequal *segments*; the squares of the unequal segments of the whole *line*, is double both of the square of half *the line*, and of the *line* between the *points* of section. Which was to be demonstrated.

PROP. X.

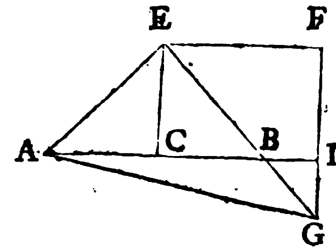
If a straight line be cut in halves, and any straight line be added to it in a straight line; the square of the whole with the part produced, and the square of the part produced, these *squares* taken both together are double both of the square of half the line, and of the square described upon the compounded *line*, that is of half *the line* and the part produced, as upon one *line*.

For let any straight line AB be cut in halves at the *point* C; and let any straight line BD be added to it in a straight line; I say that the squares of AD, DB are double of the squares of AC, CD.

For from the point C let CE be drawn at right angles to AB (by 11. 1.) and let it be made equal to either of the lines AC, CB; and let AE, EB be joined; and through the *point* E let EF be drawn (by 31. 1.) parallel to AD; and through the *point* D let DF be drawn parallel to CE: And because a certain straight line EF hath fallen upon the parallel straight lines EC, FD; (by 29. 1.) the angles CEF, EFD are therefore equal to two right angles; therefore the angles FEB, EFD are less than two right angles; but (by com. not. 11.) straight lines produced from *angles* less than two right angles meet *one another*; therefore EB, FD being produced towards the parts B, D will meet *one another*; let them be produced and meet at the *point* G; and let AG be joined.

And

Book II.



And becaufe AC is equal to CE, (by 5. 1.) the angle AEC is equal to the *angle* EAC; and the *angle* at C is a right angle; wherefore each of the angles EAC, AEC is half a right angle. Certainly for the fame reafon alfo, each of the angles CEB, EBC is half a right angle; wherefore the angle AEB is a right angle; and fince EBC is half a right angle; therefore alfo (by 15. 1.) DBG is half a right angle; but (by 29. 1.) BDG is a right angle, for it is equal to DCE, for *they are* alternate; wherefore the remaining angle DGB is (by 32. 1.) half a right angle; therefore the angle DGB is equal to the angle DBG; fo that alfo (by 6. 1.) the fide BD is equal to the fide GD: Again becaufe EGF is half a right angle; and the angle at F (by 34. 1.) is a right angle, for it is equal to the oppofite angle at C: wherefore the remaining angle FEG is half a right angle; wherefore the angle EGF is equal to the angle FEG; fo that (by 6. 1.) the fide GF is equal to the fide EF. And becaufe EC is equal to CA; alfo the fquare of EC is equal to the fquare of AC; therefore the fquares of EC, CA are double of the fquare of CA; but (by 47. 1.) the *fquare* of EA is equal to the *fquares* of EC, CA; therefore the fquare of EA is double of the fquare of AC. Again, becaufe GF is equal to EF, alfo the fquare of FG is equal to the fquare of EF; therefore the *fquares* of FG, FE are double of the *fquare* of FE; but the fquare of EG is equal to the fquares of EF, FG; wherefore the *fquare* of EG is double of the fquare of EF; but EF is (by 34. 1.) equal to CD; therefore the fquare of EG is double of the *fquare* of CD; but the *fquare* of EA has been alfo demonftrated to be double of the *fquare* of AC; wherefore the fquares of AE, EG are double of the fquares of AC, CD: but the fquare of AG is (by 47. 1.) equal to the fquares of AE, EG; therefore the *fquare* of AG is double of the *fquares* of AC, CD: but the *fquare* of AG is equal to the *fquares* of AD, DG; wherefore the fquares of AD, DG; are double of the fquares of AC, CD: but DG is equal to DB; therefore the fquares of AD, DB are double of the fquares of AC, CD.

 Where-

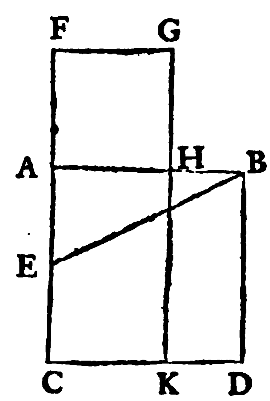
Book II. Wherefore if a ſtraight line be cut in halves, and any ſtraight line be added to it in a ſtraight line; the ſquare of the whole with the part produced, and the ſquare of the part produced, theſe *ſquares* taken both together are double both of the ſquare of half the line, and of the ſquare deſcribed upon the compounded *line*, that is of half *the line* and the part produced, as upon one *line*. Which was to be demonſtrated.

PROP. XI.

To cut a given ſtraight line, ſo that the rectangle contained by the whole and one of the ſegments may be equal to the ſquare of the remaining ſegment.

Let AB be the given ſtraight line; it is required to cut the *line* AB, ſo that the rectangle contained by the whole and one of the ſegments, may be equal to the ſquare of the remaining ſegment.

For let the ſquare ABDC be deſcribed upon AB; and let AC be cut in halves in the point E; and let BE be joined; and let CA be produced to F; and let EF be made equal to BE; and let the ſquare HF be deſcribed upon AF; and let GH be produced to K; I ſay that AB is cut in the point H ſo that the rectangle contained by AB, BH is equal to the ſquare of AH.



For becauſe the ſtraight line AC is cut in halves in the *point* E, and AF is added to it; therefore the rectangle contained by CF, FA together with the ſquare of AE is equal to the ſquare of EF; but EF is equal to EB; wherefore the rectangle contained by CF, FA together with the ſquare of AE is equal to the ſquare of EB; but (by 47.1.) the *ſquares* of BA, AE are equal to the *ſquare* of EB; for the angle at A is a right angle; therefore the *rectangle contained* by CF, FA together with the ſquare of AE is equal to the *ſquares* of BA, AE; let the common ſquare of AE be taken away; therefore the remainder, the *rectangle contained* by CF, FA is equal to the ſquare of AB; and the *rectangle* FK is the *rectangle contained* by CF, FA; for FA is

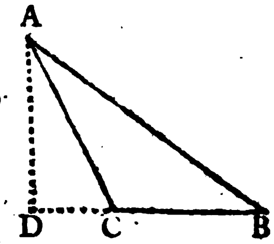
is equal to FG; and AD *is the ſquare* of AB; therefore FK is equal to AD; let AK *which is* common be taken away; therefore the remainder FH is equal to HD; and HD is the rectangle contained by AB, BH; for AB is equal to BD: and FH is the ſquare of AH; wherefore the rectangle contained by AB, BH is equal to the ſquare of AH. Book II.

Wherefore the given ſtraight line AB hath been cut in the *point* H, ſo that the rectangle contained by AB, BH is equal to the ſquare of HA. Which was to be done.

PROP. XII.

In obtuſe angled triangles, the ſquare, of the ſide ſubtending the obtuſe angle, is greater than the ſquares of the ſides containing the obtuſe angle by the *rectangle* twice *taken* contained by one of the *ſides* about the obtuſe angle upon which produced the perpendicular falls, and the *line* intercepted without, from the perpendicular to the obtuſe angle.

Let ABC be an obtuſe angled triangle, having an obtuſe angle, the *angle* ACB; and let the perpendicular AD be drawn from the point A, upon BC produced: I ſay that the ſquare of AB is greater than the ſquares of AC, CB by the rectangle contained by BC, CD *taken* twice.



For ſince the ſtraight line BD hath been cut as it may happen in the point C; wherefore (by 4. 2.) the *ſquare* of BD is equal to the ſquares of BC, CD and the rectangle contained by BC, CD *taken* twice: let the common ſquare of AD be added; therefore the *ſquares* of BD, DA are equal to the ſquares of BC, CD, DA and to the rectangle contained by BC, CD *taken* twice: but the *ſquare* of AB is (by 47. 1.) equal to the *ſquares* of BD, DA; for the angle at D is a right angle; and the *ſquare* of AC is equal to the *ſquares* of CD, DA; wherefore the *ſquare* of AB is equal to the ſquares of BC, CA and to the rectangle contained by BC, CD *taken* twice; ſo that the ſquare of AB is greater than

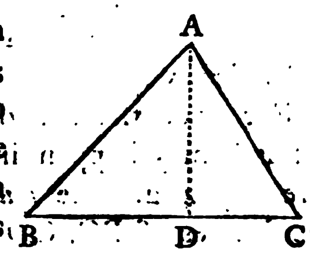
Book II. than the squares of BC, CA by the rectangle contained by BC, CD *taken* twice.

Wherefore in obtuse angled triangles the square of the side subtending the obtuse angle is greater than the squares of the sides containing the obtuse angle by the *rectangle* twice *taken* contained by one of *the sides* about the obtuse angle upon which produced the perpendicular falls, and the *line* intercepted without from the perpendicular to the obtuse angle. Which was to be demonstrated.

PROP. XIII.

In acute angled triangles, the square of the side subtending the acute angle is less than the squares of the sides containing the acute angle, by the rectangle *taken* twice, contained by one of the *sides* about the acute angle upon which the perpendicular falls and the *line* intercepted within, from the perpendicular to the acute angle.

Let there be an acute angled triangle, the *triangle* ABC, having the *angle* at B an acute angle; and let the perpendicular AD be drawn from the point A upon BC: I say that the square of AC is less than the squares of CB, BA by the rectangle CB, BD *taken* twice.



For, because the straight line CB hath been cut as it may happen in the *point* D; therefore the squares of CB, BD are equal to the rectangle CB, BD *taken* twice and to the square of DC (by 7. 2.): let the common square of AD be added: therefore the squares of CB, BD, DA are equal to the rectangle contained by CB, BD *taken* twice, and to the squares of CD, DA: but the *square* of AB is equal to the *squares* of BD, DA (by 47. 1.), for the angle at D is a right angle; and the *square* of AC is equal to the *squares* of CD, DA: wherefore the *squares* of CB, BA are equal to the rectangle contained by CB, BD, *taken* twice and the square of AC; so that the *square* of AC alone is less than the squares of CB, BA by the rectangle contained by CB, BD *taken* twice.

Where-

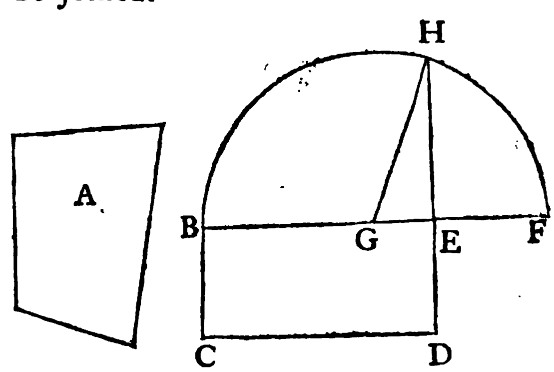
Wherefore in acute angled triangles, the ſquare of the ſide ſubtending the acute angle is leſs than the ſquares of the ſides containing the acute angle, by the rectangle *taken* twice, contained by one of the *ſides* about the acute angle upon which the perpendicular falls, and the line intercepted within, from the perpendicular to the acute angle. Which was to be demonſtrated. Book II.

PROP. XIV.

To make a ſquare equal to a given rectilineal figure.

Let A be the given rectilineal *figure*; it is required to make a ſquare equal to the given rectilineal *figure* A.

Let the right angled parallelogram BD be made (by 45. 1.) equal to the rectilineal *figure* A; if therefore BE be equal to ED what was required hath been done; for the ſquare BD is made equal to the rectilineal *figure* A: but if not, one of them BE, ED is greater: let BE be the greater, and let it be produced to F, and make EF equal to ED; and let FB be cut in halves (by 10. 1.) in the *point* G; and with the center G and at the diſtance of one of the lines GB, GF let the ſemicircle BHF be deſcribed, and let DE be produced to H, and let GH be joined.



Therefore becauſe the ſtraight line EF hath been cut into equal *ſegments* at the point G, and into unequal *ſegments* at E; therefore (by 5. 2.) the rectangle contained by BE, EF together with the ſquare of EG is equal to the ſquare of GF; but GF is equal to GH; therefore the *rectangle contained* by BE, EF together with the *ſquare* of EG is equal to the ſquare of GH; but the ſquares of HE, EG (by 47. 1.) are equal to the ſquare of GH; wherefore the *rectangle contained* by BE, EF together with the *ſquare* of GE is equal to the *ſquares* of HE, EG: let the common ſquare of EG be taken away; therefore what remains, the rectangle contained

Book II. tained by BE, EF is equal to the ſquare of EH: but the rectangle contained by BE, EF is BD, for EF is equal to ED: therefore the parallelogram BD is equal to the ſquare of HE; but BD is equal to the rectilineal *figure* A: therefore the rectilineal *figure* A is equal to the ſquare deſcribed upon EH.

Wherefore a ſquare is made equal to the given rectilineal *figure* A, the ſquare deſcribed upon EH. Which was to be done.

THE
ELEMENTS
OF
EUCLID.

BOOK III.

DEFINITIONS.

1. EQUAL circles, are those of which the diameters are equal; or of which the *straight lines* from the centers are equal. Book III.

2. A straight line is said to touch a circle, which meeting the circle, and being produced does not cut the circle.

3. Circles are said to touch one another, which, meeting each other, do not cut one another.

4. In a circle straight lines are said to be equally distant from the center, when the perpendiculars drawn from the center upon them are equal. 5. But *that line* is said to be more distant upon which the greater perpendicular falls.

6. A segment of a circle is the figure bounded by a straight line and the circumference of a circle. 7. But an angle of a segment is that contained by a straight line and the circumference of a circle.

Book III. 8. But an angle in a segment is; when any point is taken in the circumference of the segment, and from it straight lines are joined to the extremities of the straight line which is the base of the segment; the angle contained by the straight lines so joined. 9. But when the straight lines containing the angle receive any circumference, the angle is said to stand upon that. 10. And a sector of a circle is, when an angle stands at the center of the circle, the figure bounded by the straight lines containing the angle, and the circumference intercepted by them.

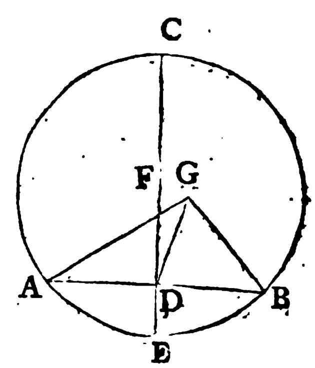
11. Similar segments of a circle are such as receive equal angles; or in which the angles are equal to one another.

PROP. I.

To find the center of a given circle.

Let the given circle be *the circle* ABC: it is required to find the center of the circle.

Let any straight line AB be drawn in it as it may happen; and let it be cut in halves at the *point* D, and from the point D let DC be drawn at right angles to AB, and let it be produced to E; and let CE be cut in halves at the *point* F: I say that F is the center of the circle ABC.



For if not, but if possible let it be G; and let GA, GD, GB be joined; and because AD is (by const.) equal to DB, and DG common; certainly the two AD, DG are equal to the two GD, DB each to each; and the base GA is (by def. 15.1.) equal to the base GB; for they are from the center G; wherefore the angle ADG is equal to the angle GDB; but when a straight line standing upon a straight line makes the adjacent angles equal to one another, each of the equal angles is a right angle (by def. 10. 1.); therefore GDB is a right angle; but FDB is also a right angle; wherefore FDB is equal to GDB, the greater to the less, which is impossible; therefore the *point* G is not the center of

of the circle ABC: certainly in the ſame manner we ſhall demonſtrate that neither is any other but the *point* F. Book III.

Wherefore the point F is the center of the circle. Which was to be done.

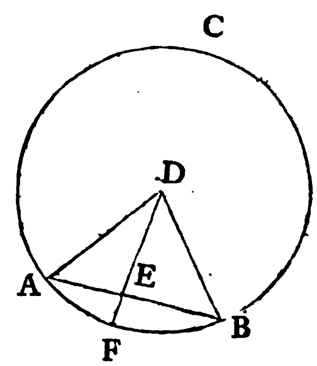
Cor. Certainly from this it is manifeſt, that if in a circle any ſtraight line, cut any ſtraight line in halves and at right angles, the center of the circle is in the cutting *line*.

PROP. II.

If two accidental points be taken in the circumference of a circle; the ſtraight line joining theſe points will fall within the circle.

Let ABC be a circle, and let two accidental points be taken in the circumference of it, *as the points* A, B; I ſay that the ſtraight line, joining A and B, will fall within the circle.

For if not, but if poſſible let AEB fall without; and let the center of the circle ABC be taken (by 1. 3.), and let it be D; and let AD, DB be joined; and let DF be produced to E.



And ſince DA is equal to DB, therefore the angle DAE (by 5. 1.) is equal to DBE; and becauſe one ſide of the triangle DAE is viz. produced the ſide AEB, therefore (by 16. 1.) the angle DEB is greater than DAE: but DAE is equal to DBE; therefore DEB is greater than DBE; but (by 19. 1.) the greater ſide is extended under the greater angle; therefore DB is greater than DE; but DB is equal to DF; therefore DF is greater than DE; the leſs than the greater, *ſuppoſing E without the circle*, which is impoſſible: therefore the ſtraight line joining A and B will not fall without the circle: certainly in the ſame manner we ſhall demonſtrate that neither will it fall in the circumference: therefore within *the circle*.

Wherefore if two accidental points be taken in the circumference of a circle, the ſtraight line, joining theſe points, will fall within the circle. Which was to be demonſtrated.

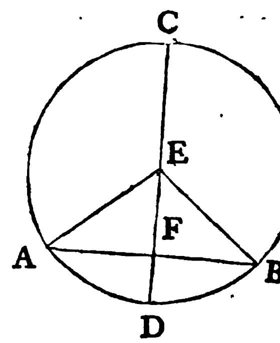
Book III.

PROP. III.

If in a circle any ſtraight line through the center cut in halves any ſtraight line not through the center, it will alſo cut it at right angles; and if it cut it at right angles it will alſo cut it in halves.

Let ABC be a circle, and let CD any ſtraight line in it through the center cut in halves AB any ſtraight line, not through the center, in the point F; I ſay that it alſo cuts it at right angles.

For let the center of the circle ABC be taken (by 1. 3.); and let it be E, and let EA, EB be joined.



And becauſe AF is equal to FB, and FE common; certainly the two are equal to the two; and the baſe EA is equal to the baſe EB; and (by 8. 1.) the angle AFE is equal to the angle BFE: but when a ſtraight line ſtanding upon a ſtraight line makes the adjacent angles equal to one another, each of the equal angles is a right angle; therefore each of the angles AFE, BFE is a right angle: therefore CD through the center cutting in halves AB not paſſing through the center, alſo cuts it at right angles.

But let CD cut AB at right angles; I ſay that it alſo cuts it in halves; that is, that AF is equal to FB.

For the ſame things being conſtructed, becauſe EA from the center is equal to EB, the angle alſo EAF is equal (by 5. 1.) to EBF; but the right angle AFE is equal to the right angle BFE; therefore there are two triangles EAF, EBF having the two angles equal to the two angles, and one ſide equal to one ſide, viz. EF common to them, extended under one of the equal angles; they will therefore alſo have (by 26. 1.) the remaining ſides equal to the remaining ſides; wherefore AF is equal to BF.

Wherefore if in a circle, any ſtraight line through the center cut in halves any ſtraight line not through the center; it will alſo cut it at right angles; and if it cut it at right angles it will alſo cut it in halves. Which was to be demonſtrated.

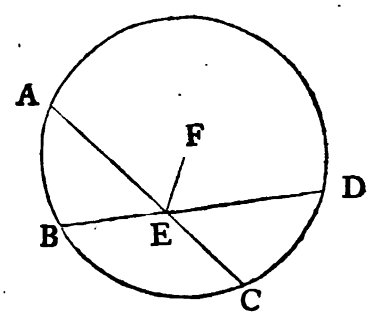
PROP.

PROP. IV.

If in a circle two ſtraight lines cut one another, not paſſing through the center; they do not cut one another in halves.

Let ABCD be a circle, and let two ſtraight lines in it AC, BD cut one another in the point E, not paſſing through the center; I ſay that they do not cut one another in halves.

For if *it be* poſſible, let them cut one another in halves, ſo that AE is equal to EC and BE to ED; and let the center of the circle ABCD be taken (by 1. 3.), and let it be F; and let FE be joined.



Wherefore ſince a certain ſtraight line FE through the center cuts in halves a certain ſtraight line AC not drawn through the center, it will alſo cut it (by 3.3.) at right angles; therefore FEA is a right angle: Again becauſe a certain ſtraight line FE cuts in halves a certain ſtraight line BD, not *paſſing* through the center, it will alſo (by 3. 3.) cut it at right angles; wherefore FEB is a right angle; but FEA has been alſo demonſtrated to be a right angle: wherefore the *angle* FEA is equal to the *angle* FEB; the leſs to the greater which is impoſſible: wherefore the ſtraight lines AC, BD do not cut one another in halves.

Wherefore if in a circle two ſtraight lines cut one another, not paſſing through the center; they do not cut one another in halves. Which was to be demonſtrated.

PROP. V.

If two circles cut one another, they will not have the ſame center.

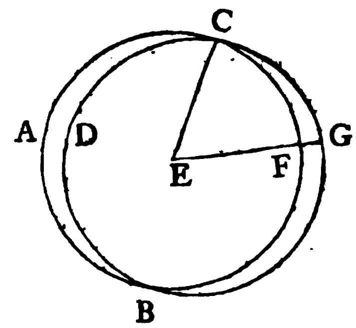
For let two circles ABC, CDG cut one another in the points, B, C; I ſay they will not have the ſame center.

For if *it be* poſſible, let it be E; and let EC be joined; and let EFG be drawn as it may happen.

And becauſe the point E is the center of the circle ABC, EC is equal to EF: again becauſe the point E is the center of the circle CDG, EC is equal to EG; but EC has been already demonſtrated

to

Book III. *to be* equal to EF: wherefore EF is equal to EG, the less to the greater, which is impossible: wherefore the point E is not the center of the circles ABC, CDG.



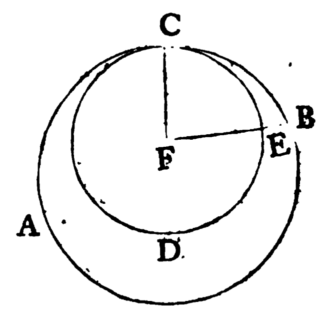
Therefore if two circles cut one another, they will not have the same center. Which was to be demonstrated.

PROP. VI.

If two circles touch one another within, they will not have the same center.

For let the two circles ABC, CDE touch one another in the point C; I say that they will not have the same center.

For if *it be* possible, let it be F, and join FC, and let FEB be drawn as it may happen.



Wherefore since the point F is the center of the circle ABC; FC is equal to FB; again, because the point F is the center of the circle CDE; FC is equal to FE; but FC has been also demonstrated *to be* equal to FB; therefore FE is equal to FB, the less to the greater, which is impossible: wherefore the point F is not the center of the circles ABC, CDE.

Wherefore if two circles touch one another within, they will not have the same center. Which was to be demonstrated.

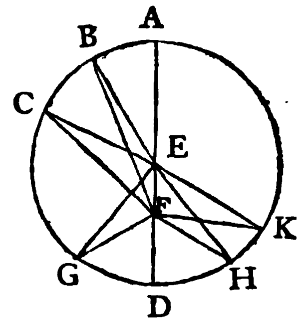
PROP. VII.

If any point be taken in the diameter of a circle, which is not the center of the circle; and from this poiut let certain straight lines fall upon the circle, that will be the greatest in which the center *is*; and the remainder *of the diameter* the least; but of the others, the nearer to the line through the center is always greater than one more remote; and two equal straight lines only will fall from the same point upon the circle, on each side of the least.

Let

Let ABCD be a circle, and let AD be its diameter; and let any point F be taken in AD, which is not the center of the circle; and let E be the center of the circle; and from the *point* F let certain ſtraight lines FB, FC, FG fall upon the circle ABCD: I ſay that FA is the greateſt, and FD the leaſt; but of the others FB is greater than FC; and FC than FG. Book III.

For let BE, CE, GE be joined.



And becauſe the two ſides of every triangle are greater than the remaining *ſide*; therefore EB, EF are greater than BF; but AE is equal to BE; wherefore BE, EF are equal to AF: therefore AF is greater than BF: Again, becauſe BE is equal to CE, and FE common; certainly the two BE, EF are equal to the two CE, EF; but the angle BEF is alſo greater than the angle CEF; wherefore (by 24. 1.) the baſe BF is greater than the baſe CF. Certainly for the ſame reaſon alſo CF is greater than FG.

Again, ſince GF, FE are (by 20. 1.) greater than EG; and EG is equal to ED; therefore GF, FE are greater than ED; Let EF *which is* common be taken away; therefore the remainder GF is greater than the remainder FD; wherefore FA is the greateſt, and FD the leaſt; and FB is greater than FC; and FC than FG.

I ſay that alſo only two equal ſtraight lines will fall from the point F upon the circle ABCD, *one* on each *ſide* of the leaſt FD: for let the *angle* FEH be made with the ſtraight line EF, and at the point E in it, equal to the angle GEF; and let FH be joined wherefore becauſe GE is equal to EH, and EF common; certainly the two GE, EF are equal to the two HE, EF; and the angle GEF is equal to the angle HEF; wherefore (by 4. 1.) the baſe FG is equal to the baſe FH: I ſay that another *line* equal to FG will not fall upon the circle, from the point F: for if it be poſſible, let FK fall; and ſince FK is equal to FG, but FH is equal to FG; therefore FK is equal FH; the nearer to the *line* through the center equal to the *one* more remote; which is impoſſible.

OR

Book III. OR THUS. Let EK be joined; and becaufe GE is equal to EK, and FE common, and the bafe GF equal to the bafe FK; therefore (by 8. 1.) the angle GEF is equal to the angle KEF; but the angle GEF is equal to the *angle* HEF; and therefore the *angle* HEF is equal to the *angle* KEF; the lefs to the greater which is impoffible: therefore no other line will fall from the point F upon the circle equal to GF: therefore one only.

Wherefore if any point be taken in the diameter of a circle &c. Which was to be demonftrated.

PROP. VIII.

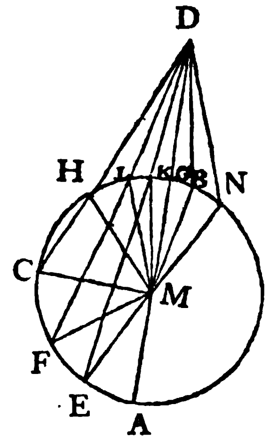
If any point be taken without a circle, and certain ftraight lines be drawn from the point to the circle, one of which *paffes* through the center, but the other *lines* as it may happen; of thofe ftraight lines falling upon the concave circumference, the *one paffing* through the center is the greateft: but of the other *lines*, the *one* nearer the *liue paffing* through the center will always be greater than that more remote: But of the ftraight lines falling upon the convex circumference, the leaft is that between the point and the diameter: but of the other lines the nearer to the leaft is lefs than that more remote; and two equal ftraight lines only will fall upon the circle from the point, *one* on each *fide* of the leaft.

Let ABC be a circle; and let any point D be taken without the circle ABC; and let certain ftraight lines be drawn from it to the circle viz. DA, DE, DF, DC; and let AD be through the center; I fay that of the ftraight lines falling upon AEFC the concave circumference, the greateft is DA the *line* through the center: but the *line* nearer to the *one* through the center will be greater than *one* more remote; viz DE than DF and DF *greater* than DC; but of the ftraight lines falling upon the convex circumference HLKG, DG is the leaft, which is between the point D and the diameter AG; but the nearer to the leaft DG is lefs than one more remote; viz. DK *lefs* than DL and DL than DH.

For let the center of the circle ABC be taken; and let it be M; and let ME, MF, MC, MK, ML, MH be joined.

And

And becaufe AM is equal to EM; let MD *which is* common be added; therefore AD is equal to EM, MD: but (by 20. 1.) EM, MD are greater than ED; and therefore AD is greater than ED: Again becaufe ME is equal to MF; and MD common; therefore EM, MD are equal to MF, MD; and the angle EMD is greater than the angle FMD; wherefore (by 24. 1.) the bafe ED is greater than the bafe FD: certainly in the fame manner we fhall demonftrate alfo that DF is greater than CD: wherefore AD is the greateft; and DE greater than DF; and DF than DC.



And fince MK, KD are greater than MD; and MG equal to MK, therefore the remainder KD is greater than the remainder GD; fo that GD is lefs than KD, wherefore it is the leaft. And becaufe upon MD one of the fides of the triangle MLD two ftraight lines MK, KD are joined together within the triangle; therefore (by 21. 1.) MK, KD are lefs than ML, LD; of which MK is equal to ML; therefore the remainder DK is lefs than the remainder DL: certainly in the fame manner we fhall demonftrate that DL is lefs than DH: wherefore DG is the leaft; and DK is lefs than DL; and DL than DH.

Alfo I fay that only two equal ftraight lines will fall from the point D upon the circle; *one* on each fide of DG the leaft: with the ftraight line MD, and at the point M in it, let the angle DMB be made equal to the angle KMD; and let DB be joined; and becaufe MK is equal to MB, and MD common; certainly the two KM, MD are equal to the two BM, MD, each to each; and the angle KMD is equal to the angle BMD: wherefore (by 4. 1.) the bafe DK is equal to the bafe DB: I fay that another *ftraight line* equal to the ftraight line DK will not fall upon the circle from the point D: for if it be poffible let it fall; and let it be DN; becaufe therefore DK is equal to DN; but DK is equal to DB; therefore alfo DB is equal to DN; the nearer to the leaft DG equal to *one* more remote, which has been fhewn to be impoffible.

Or otherwise. Let MN be joined, and becaufe KM is equal

Book III. to MN, and MD common, alſo the baſe DK equal to the baſe DN; wherefore the angle KMD is equal to the angle DMN; but the *angle* KMD is equal to BMD: therefore alſo BMD is equal to NMD; the leſs to the greater, which is impoſſible: Therefore more than two ſtraight lines will not fall upon the circle ABC from the point D, *one* on each *ſide* of the leaſt GD.

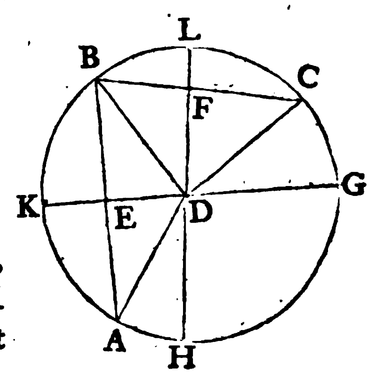
Wherefore if any point be taken without a circle &c. Which was to be demonſtrated.

PROP. IX.

If any point be taken within a circle, and more than two equal ſtraight lines fall from the point upon the circle: the point taken is the center of the circle.

Let ABC be a circle, and the point D within it; and let DA, DB, DC; more than two equal ſtraight lines, fall from the *point* D to the circle ABC: I ſay that the point D is the center of the circle ABC.

For let AB, BC be joined, and let them be cut in halves in the points E, F; ED, DF being joined; let them be produced to the points G, K, H, L.

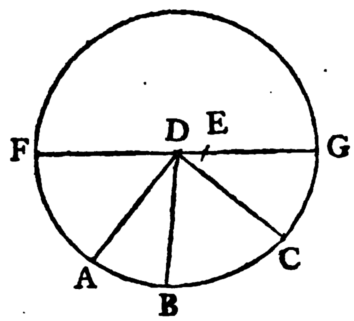


Wherefore becauſe AE is equal to EB, and ED common: certainly the two AE, ED are equal to the two BE, ED; and the baſe DA is equal to the baſe DB; therefore (by 8. 1.) the angle AED is equal to the angle BED; wherefore each of the angles AED, BED is a right angle; wherefore GK cutting AB in halves cuts it alſo at right angles; and becauſe (by cor. prop. 1. 3.) if in a circle any ſtraight line cut any ſtraight line in halves and at right angles, the center of the circle is in the cutting line; therefore the center of the circle ABC is in GK: certainly for the ſame reaſon alſo the center of the circle ABC is in HL; and the ſtraight lines GK, HL have no other *point* in common, but the point D; therefore the point D is the center of the circle ABC.

Where-

Wherefore if any point be taken within a circle; and more than two equal ſtraight lines fall from the point upon the circle; the point taken is the center of the circle. Which was to be demonſtrated. Book III.

OTHERWISE. For let there be, any point D, taken within the circle ABC and let more than two equal ſtraight lines DA, DB, DC fall from the *point* D upon the circle ABC; I ſay that D the point taken is the center of the circle ABC.

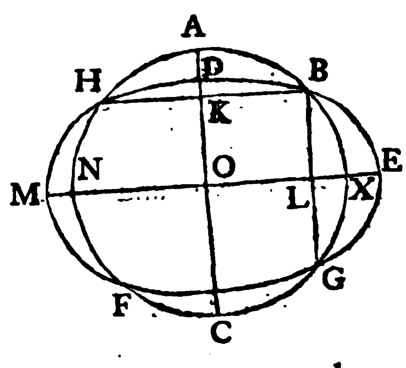


For if not; but if poſſible let it be E; and DE being joined, let it be produced to the points F, G; but FG is a diameter of the circle ABC: therefore becauſe a certain point hath been taken in FG the diameter of the circle ABC, which is not the center of the circle, viz. the *point* D; DG will be the greateſt (by prop. 7. 3.) and DC greater than DB; and DB than DA; but *they are* alſo equal; which is impoſſible: wherefore E is not the center of the circle ABC: certainly in the ſame manner we ſhall demonſtrate, that neither is any other but the *point* D: wherefore the point D is the center of the circle ABC.

PROP. X.

A circle does not cut a circle in more points than two.

For if *it be* poſſible; let the circle ABC cut the circle DEF in more points than two, viz. in B, G, H; and BG, BH being joined, let them be cut in halves in the points K, L; and from the points K, L having drawn KC, LM at right angles to BG, BH let them be produced to the points A, E.



Wherefore becauſe in the circle ABC a certain ſtraight line AC cuts in halves a certain ſtraight line BH and at right angles; therefore the center of the circle ABC is in the ſtraight line AC: again, becauſe in the ſame circle ABC a certain ſtraight line NX cuts in halves and at right angles a certain ſtraight line BG;

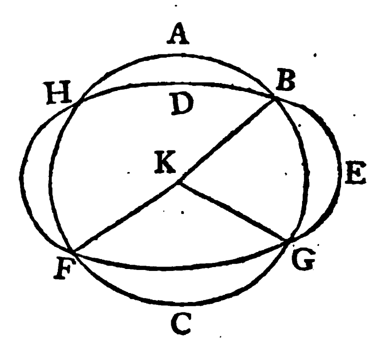
 there-

Book III. therefore the center of the circle ABC is in NX; but it has alſo been demonſtrated to be in AC; and the ſtraight lines AC, NX meet one another in no *point* but the *point* O; therefore the point O is the center of the circle ABC: certainly in the ſame manner we ſhall demonſtrate that O is the center of the circle DEF; therefore two circles ABC, DEF cutting one another have the ſame center the *point* O which is impoſſible (by 5. 3.).

Wherefore a circle does not cut a circle in more points than two. Which was to be demonſtrated.

Otherwise. For again, let the circle ABC cut the circle DEF in more points than two; viz. in the *points* B, G, F; and let the center of the circle ABC be taken the *point* K; and let KF, KG, KB be joined.

Wherefore ſince a certain point hath been taken within the circle DEF, the point K; and from K more than two equal ſtraight lines KB, KF, KG have fallen upon the circle DEF: therefore (by 9. 3.) the point K is the center of the circle DEF; but K is alſo the center of the circle ABC; therefore two circles cutting one another have the ſame center K; which (by 5. 3.) is impoſſible.



Wherefore a circle does not cut a circle in more points than two. Which was to be demonſtrated.

PROP. XI.

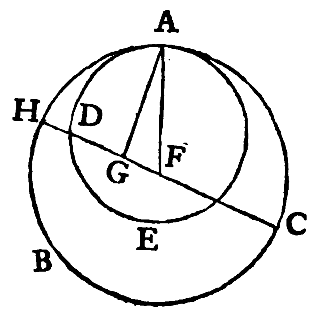
If two circles touch one another inwardly, and their centers be taken; the ſtraight line joining the centers of them, being produced, will fall upon the contact of the circles.

For let two circles ABC, ADE touch one another inwardly at the point A; and let F the center of the circle ABC be taken, and G the *center* of the *circle* ADE: I ſay that the ſtraight line joining G and F being produced will fall upon the point A.

For if not, but if poſſible let it fall as the *line* FGDH; and let AF, AG be joined.

Where-

Wherefore becaufe AG, GF are greater than FA, that is than FH; for FA is equal to FH for they are both from the center; let FG which is common be taken away; therefore the remainder AG is greater than the remainder GH; but AG is equal to GD (by def. 15. 1.); therefore GD is greater than GH; the lefs than the greater; which is impoffible: wherefore the ftraight line joining F and G will not fall without the contact at A; therefore upon it.



Wherefore if two circles touch one another inwardly, and their centers be taken; the ftraight line joining the centers of them, being produced, will fall upon the contact of the circles. Which was to be demonftrated.

OTHERWISE. But let it fall as GFC; and let GFC be produced in a ftraight line to the point H; and let AG, AF be joined.

Wherefore fince AG, GF are greater (by 20. 1.) than AF; but AF is equal to CF: that is to FH; let FG *which is* common be taken away; therefore the remainder AG is greater than the remainder GH; that is GD *is greater* than GH; the lefs than the greater which is impoffible; in the fame manner we fhall fhew the fame *thing* to be abfurd, if the center of the greater circle be without the lefs.

PROP. XII.

If two circles touch one another outwardly, the ftraight line joining their centers will pafs through the contact.

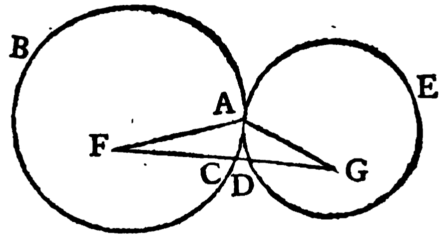
For let two circles, ABC, ADE touch one another outwardly in the point A; and let F the center of the circle ABC be taken; and G the *center* of ADE; I fay that the ftraight line joining F and G will pafs through the contact A.

For if not, but if poffible let it pafs as FCDG; and let AF, AG be joined.

Where-

Book III.

Wherefore becauſe the point F is the center of the circle ABC; FA is equal to FC: again becauſe the point G is the center of the circle ADE; AG is equal to GD; but FA has been demonſtrated to be equal to FC; therefore FA, AG are equal to FC, DG; ſo that the whole FG is greater than FA, AG; but *it is* alſo leſs (by 20. 1.); which is impoſſible: wherefore the ſtraight line joining F and G will not not paſs through the contact at A; therefore *it will paſs* through it.



Wherefore if two circles touch one another outwardly, the ſtraight line joining their centers will paſs through the contact: Which was to be demonſtrated.

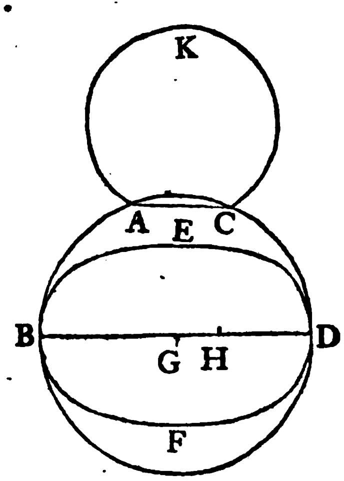
PROP. XIII.

A circle will not touch a circle in more points than one, whether it touch it within or without.

For if it be poſſible, let the circle ABDC firſt touch the circle EBFD within, in more points than one; viz. in the points, B, D.

And let G, the center of the circle ABDC be taken (by 1. 3.); and H the center of the circle EBFD.

Wherefore the ſtraight line joining G and H will (by 11. 3.) fall upon the *points* B, D: let it fall as BGHD; and ſince the point G is the center of the circle ABDC; BG is equal to GD; wherefore BG is greater than HD; therefore BH is much greater than HD; Again becauſe the point H is the center of the circle EBFD; BH is equal to HD; but it hath alſo been demonſtrated to be much greater than it; which is impoſſible; wherefore a circle does not touch a circle in more points than one inwardly.



I ſay

I ſay that neither does it outwardly. For if it be poſſible; let the circle ACK touch the circle ABDC without in more points than one, in the points A, C and let AC be joined. Book III.

Wherefore becauſe any two points A, C have been taken in the circumference of each of the circles ABDC, ACK; the ſtraight line joining theſe points will fall (by 2. 3.) within both: but it hath fallen within the circle ABDC, and without the circle ACK; which is abſurd: wherefore a circle does not touch a circle outwardly in more points than one; but it has been demonſtrated that neither *does it* inwardly.

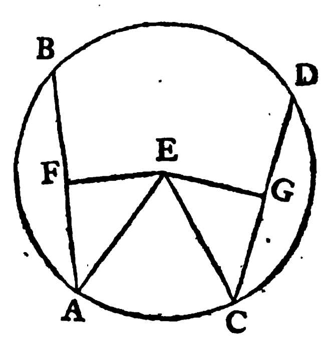
Wherefore a circle does not touch a circle in more points than one, whether it touch it on the inſide or on the outſide. Which was to be demonſtrated.

PROP. XIV.

The equal ſtraight lines, in a circle, are equally diſtant from the center; and the ſtraight lines being equally diſtant from the center are equal to one another.

Let ABDC be a circle; and let AB, CD be equal ſtraight lines in it; I ſay that they are equally diſtant from the center.

For let the center of the circle ABDC be taken (by 1. 3.); and let it be E; and let EF, EG be drawn perpendiculars from E to AB, CD; and let AE, EC be joined.



Wherefore ſince EF, a certain ſtraight line through the center, cuts a certain ſtraight line AB, not through the center, at right angles; it will cut it alſo in halves (by 3. 3.); wherefore AF is equal to FB: therefore AB is the double of AF. Certainly, for the ſame reaſon alſo, CD is double of CG; and AB is equal to CD; wherefore AF is equal to CG (by com. not. 7.): and becauſe AE is equal to EC; alſo the *ſquare* of AE *is* equal to the *ſquare* of EC: but the *ſquares* of AF, FE are equal (by 47. 1.) to the *ſquare* of AE; for the angle at F is a right angle: and the *ſquares* of EG, GC are equal to the *ſquare* of EC; for the angle

Book III. angle at G is a right angle: wherefore the *ſquares* of AF, FE are equal to the *ſquares* of CG, GE; of which the *ſquare* of AF is equal to the *ſquare* of CG; for AF is equal to CG; therefore the remainder the *ſquare* of FE is equal to the remainder the *ſquare* of EG; wherefore FE is equal to EG: but in a circle, ſtraight lines are ſaid to be equally diſtant from the center, when perpendiculars drawn from the center upon them are equal (by def. 4. 3.) therefore AB, CD are equally diſtant from the center

But let the ſtraight lines AB, CD be equally diſtant from the center, that is, let FE be equal to EG: I ſay that AB is alſo equal to CD.

For the ſame things being conſtructed; certainly we ſhall demonſtrate, in the ſame manner, that AB is the double of AF; and CD of CG: aud becauſe AE is equal to EC; the *ſquare* of AE is alſo equal to the *ſquare* of EC; but the *ſquares* of EF, FA are equal to the *ſquare* of AE; and the *ſquares* of EG, GC are equal to the *ſquare* of CE; therefore the *ſquares* of EF, FA are equal to the *ſquares* of EG, GC; of which the ſquare of EG is equal to the ſquare of EF; therefore the remaining *ſquare* of AF is equal to the remaining *ſquare* of CG; wherefore AF is equal to CG; alſo AB is the double of AF, and CD the double of CG; wherefore AB is equal to CD.

Wherefore the equal ſtraight lines, in a circle, are equally diſtant from the center; and the ſtraight lines, being equally diſtant from the center, are equal to one another. Which was to be demonſtrated.

PROP. XV.

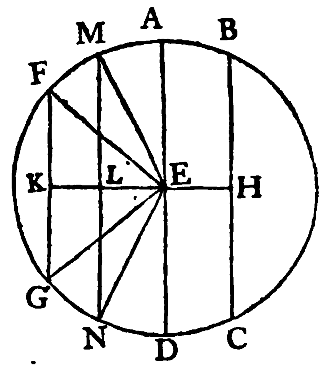
The diameter is the greateſt *line* in a circle; but of the others, the *one* nearer to the center is greater than *any one* more remote.

Let ABCD be a circle; and let AD be a diameter of it; and E the center; and let BC be nearer the center E; but FG more remote; I ſay that AD is the greateſt, and BC greater than FG.

For let EH, EK be drawn perpendiculars from the center, to BC, FG; and becauſe BC is nearer to the center, but FG more remote; therefore (by def. 5. 3.) EK is greater than EH; let EL be

be made equal to EH; and through the point L, LM being drawn Book III.
at right angles to EK let it be produced to N; and let EM, EN, EF, EG be joined.

And becaufe EH is equal to EL; (by 14. 3.) BC it equal to MN: Again fince AE is equal to EM and ED to EN; therefore AD is equal to ME, EN; but (by 20. 1.) ME, EN are greater than MN; therefore alfo AD is greater than MN: but MN is equal to BC; therefore AD is greater than BC: And becaufe the two ME, EN are equal to the two FE, EG and the angle MEN is greater than the angle FEG; therefore (by 24. 1.) the bafe MN is greater than the bafe FG; but MN has been demonftrated *to be* equal to BC; and BC is alfo greater than FG: wherefore the diameter AD *is* the greateft; and BC greater than FG.



Wherefore the diameter is the greateft *line* in a circle; but of the others the *one* nearer to the center is greater than *any one* more remote. Which was to be demonftrated.

PROP. XVI.

The *ftraight line* drawn at right angles to the diameter of a circle, from the extremity, will fall without the circle; and into the place between the ftraight line and the circumference another ftraight line will not fall; and the angle of the femicircle is greater than every acute rectilineal angle: but the remainder lefs.

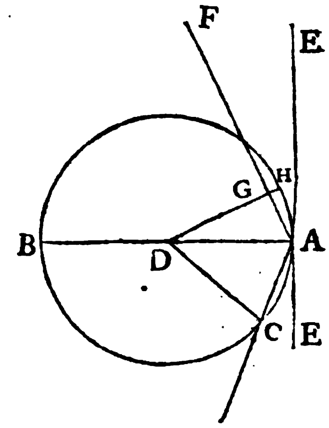
Let ABC be a circle about the center D, and AB *its* diameter; I fay that the *ftraight line* drawn from its extremity, from the point A, at right angles to AB, will fall without the circle.

For if not, but if poffible let it fall within, as AC and let DC be joined.

And becaufe DA is equal to DC, the angle DAC (by 5. 1.) is alfo equal to the angle ACD; but DAC is a right angle (by conft.) wherefore alfo ACD is a right angle, therefore the angles DAC, ACD are equal to two right angles; which (by 17. 1.) is impof-

 fible;

Book III. fible; wherefore the *ftraight line* drawn from the point A, at right angles to AB will not fall within the circle: certainly in like manner we fhall demonftrate that neither *will it fall* upon the circumference; therefore let it fall without as AE.



I fay that another ftraight line will not fall into the place, between the ftraight line AE and the circumference HAC,

For if it be poffible let it fall as FA; and let DG be drawn, from the point D, perpendicular to FA.

And becaufe AGD is a right angle, but DAG (by 17. 1.) lefs than a right angle; therefore (by 19. 1.) AD is greater than DG; but AD is equal to DH; therefore DH is greater than DG; the lefs than the greater, *fuppofing G without the circle*, which is impoffible; wherefore another ftraight line will not fall into the place between the ftraight line and the circumference.

Alfo I fay that the angle of the femicircle; the *angle* contained by the ftraight line BA and the circumference HA; is greater than every acute rectilineal angle; but the remainder, the *angle* contained by the circumference HA and the ftraight line AE is lefs than every acute rectilineal angle.

For if there be any rectilineal angle, greater than that contained by the ftraight line BA and the circumference HA; but lefs than that contained by the circumference HA and the ftraight line AE; a ftraight line will fall into the place between the circumference HA, and the ftraight line AE; which will make *an angle* contained by ftraight lines greater, than that contained by the ftraight line BA and the circumference HA; but lefs than that contained by the circumference HA and the ftraight line AE: but it does not fall: therefore, the acute angle contained by the ftraight lines will not be greater than the angle contained by the ftraight line BA and the circumference HA, nor lefs than the *angle* contained by the circumference HA and the ftraight line AE. Which was to be demonftrated.

Cor.

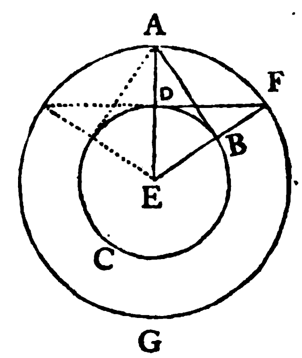
Cor. From thefe *demonftrations* it is manifeft that the *ftraight line* drawn from the extremity at right angles to the diameter of the circle, touches the circle: and that a ftraight line touches a circle in one point only: becaufe (by 2. 3.) the ftraight line meeting it in two *points* has been fhewn to fall within it. Book III.

PROP. XVII.

From a given point to draw a ftraight line touching a given circle.

Let A be the given point, and BCD the given circle; it is required from the point A to draw a ftraight line touching the circle BCD.

For let E the center of the circle be taken (by 1. 3.); and let AE be joined; and with the center E and at the diftance EA let the circle AFG be defcribed; and from the point D, let DF be drawn at right angles to AE; and let EBF, AB be joined: I fay that from the point A the *ftraight line* AB hath been drawn touching the circle BCD.



For fince E is the center of the circles BCD, AFG; therefore EA is equal to EF; and ED to EB: certainly the two AE, EB are equal to the two FE, ED; and they contain a common angle, the *angle* at E; wherefore (by 4. 1.) the bafe DF is equal to the bafe AB and the triangle DEF is equal to the triangle EBA; and the remaining angles to the remaining angles; therefore the *angle* EBA is equal to EDF; but EDF is a right angle; wherefore EBA is a right angle; and EB is from the center: but the *ftraight line* drawn from the extremity at right angles to the diameter of the circle touches the circle: wherefore AB touches the circle.

Wherefore from a given point, the *point* A, a ftraight line AB hath been drawn touching the circle BCD. Which was to be done.

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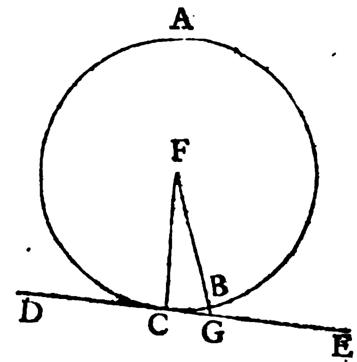
PROP. XVIII.

If any ſtraight line touch a circle, and any ſtraight line be drawn from the center to the *point of* contact; the *line* drawn will be perpendicular to the touching *line*.

Let any ſtraight line DE touch the circle ABC, in the point C; and let F the center of the circle ABC be taken; and from F to C let FC be drawn; I ſay that FC is perpendicular to DE.

For if not, let FG be drawn from the point F perpendicular to DE.



Wherefore becauſe FGC is a right angle, therefore GCF is an acute angle; but the greater ſide is extended under the greater angle; therefore FC *is* greater than FG; but FC *is* equal to FB: wherefore FB *is* greater than FG; the leſs than the greater which is impoſſible: therefore FG is not perpendicular to DE: certainly in the ſame manner we ſhall demonſtrate, that neither *is* any other but FC: wherefore FC is perpendicular to DE.

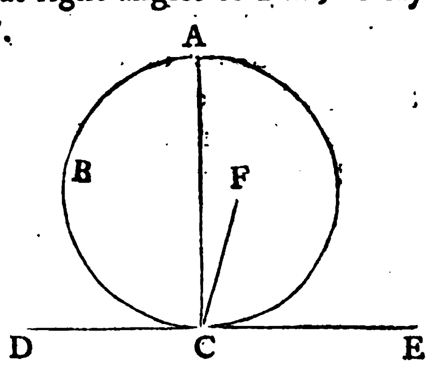
Wherefore if any ſtraight line touch a circle; and any ſtraight line be drawn from the center to the contact; the *line* drawn is perpendicular *to the tangent*. Which was to be demonſtrated.

PROP. XIX.

If any ſtraight line touch a circle, and a ſtraight line be drawn, from the contact, at right angles to the touching *line*; the center of the circle will be in the *line ſo* drawn.

For let any ſtraight line DE touch the circle ABC in the point C; and from C let CA be drawn at right angles to DE; I ſay that the center of the circle is in AC.

For if not, but if poſſible let it be F: and let CF be joined.



Wherefore becauſe a certain ſtraight line DE touches the circle ABC; and FC hath been drawn from the center to the contact; therefore FC is perpendicular to DE (by 18.3.); therefore FCE is a right angle; but (by

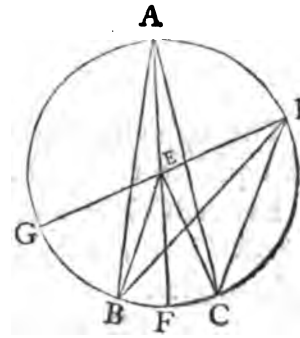
conſt.) ACE is a right angle; wherefore FCE is equal to ACE; the leſs to the greater; which is impoſſible; wherefore F is not the center of the circle ABC: certainly in the ſame manner we ſhall demonſtrate, that neither is it any other, but *ſome point* in AC.

Wherefore if any ſtraight line touch a circle, and a ſtraight line be drawn, from the contact, at right angles to the touching *line*; the center of the circle will be in the *line ſo* drawn. Which was to be demonſtrated.

PROP. XX.

In a circle, the angle at the center is double of the angle at the circumference; when the angles have the ſame circumference *for a* baſe.

Let ABC be a circle, and let BEC be an angle at the center of it; but BAC at the circumference *of it*; and let them have the ſame circumference BC *for a* baſe; I ſay that the angle BEC is double of the angle BAC.



For AE being joined let it be produced to F.

Wherefore becauſe EA is equal to EB; alſo (by 5. 1.) the angle EAB is equal to EBA; wherefore the angles EAB, EBA are double of EAB: but (by 32. 1.) BEF is equal to EAB, EBA; wherefore BEF is double of EAB: Certainly for the ſame reaſon FEC is double of EAC: therefore the whole BEC is double of the whole BAC.

Let it be bent in a different direction; and let the other angle be BDC; and DE being joined let it be produced to G: certainly in the ſame manner we ſhall demonſtrate that the angle GEC is double of the *angle* GDC; of which GEB is double of GDB; therefore the remainder BEC is double of the remainder BDC.

Wherefore in a circle, the angle at the center is double of the angle at the circumference; when the angles have the ſame circumference *for* a baſe. Which was to be demonſtrated.

PROP.

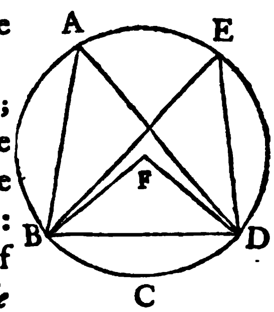
Book III.

PROP. XXI.

In a circle, the angles in the ſame ſegment are equal to one another.

Let ABCD be a circle, and let the angles BAD, BED be in the ſegment BAED; I ſay that the angles BAD, BED are equal to one another.

For let the center of the circle ABCD be taken; and let it be F; and join BF, FD.



And becauſe the angle BFD is at the center; but BAD at the circumference, and have the ſame circumference BCD *for* a baſe; therefore the angle BFD is double of the angle BAD: certainly for the ſame reaſon BFD is double of BED: wherefore (by com. not. 7.) the *angle* BAD is equal to BED.

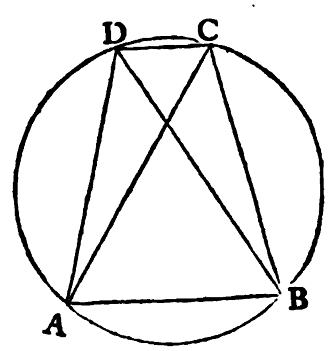
Wherefore in a circle, the angles in the ſame ſegment are equal to one another. Which was to be demonſtrated.

PROP. XXII.

The oppoſite angles, of quadrilateral *figures inſcribed* in circles, are equal to two right angles.

Let ABCD be a circle; and let ABCD be a quadrilateral *figure inſcribed* in it; I ſay that the oppoſite angles of it are equal to two right angles.

Let AC, BD be joined.



And becauſe (by 32. 1.) the three angles of every triangle are equal to two right angles; CAB, ABC, BCA the three angles of the triangle ABC are equal to two right angles: but CAB is equal to BDC (by 21. 3.); for they are in the ſame ſegment BADC: and ACB is equal to ADB (by 21. 3.): for they are in the ſame ſegment ADCB; wherefore the whole *angle* ADC is equal to the *two angles* BAC, ACB; let ABC *which is* common be added; therefore

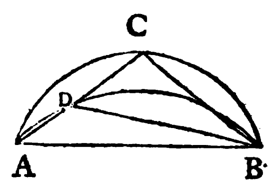
fore the *three angles* ABC, BAC, ACB are equal to the *two* ABC, ADC: but the angles ABC, BAC, ACB are equal to two right angles; alſo the angles ABC, ADC are equal to two right angles; Certainly in the ſame manner we ſhall demonſtrate, that the angles BAD, DCB are equal to two right angles. Book III.

Wherefore the oppoſite angles, of quadrilateral *figures inſcribed* in circles, are equal to two right angles. Which was to be demonſtrated.

PROP. XXIII.

Two ſimilar and unequal ſegments of circles will not ſtand upon the ſame ſtraight line, towards the ſame parts.

For if it be poſſible; let ACB, ADB two ſimilar and unequal ſegments of circles ſtand upon the ſame ſtraight line AB, towards the ſame parts, and let ADC be drawn; and let CB, DB be joined.



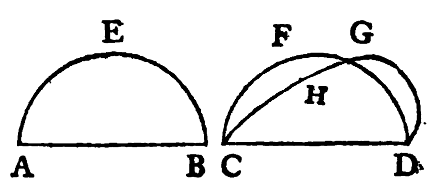
Wherefore becauſe the ſegment ACB is ſimilar to the ſegment ADB: and ſimilar ſegments of circles are (by def. 11. 3.) thoſe receiving equal angles; therefore the angle ACB is equal to the *angle* ADB; the outward to the inward; which (by 16. 1.) is impoſſible.

Wherefore two ſimilar and unequal ſegments of circles will not ſtand upon the ſame ſtraight line, towards the ſame parts. Which was to be demonſtrated.

PROP. XXIV.

The ſimilar ſegments of circles, upon equal ſtraight lines, are equal to one another.

Let AEB, CFD be ſimilar ſegments of circles upon the equal ſtraight lines AB, CD; I ſay that the ſegment AEB is equal to the ſegment CFD.



For the ſegment AEB being applied to the *ſegment* CFD; and the point A being placed upon the *point* C; and the ſtraight line AB upon CD; the point B will apply itſelf to the point D; becauſe AB is equal to CD; but the ſtraight

Book III. ſtraight line AB applying itſelf to CD, the ſegment AEB (by 23. 3.) will apply itſelf to CFD : for if the ſtraight line AB ſhall apply itſelf to CD; but the ſegment AEB will not apply itſelf to CFD; but ſhall change its direction as CHGD: but a circle does not cut a circle in more points than two (by 10. 3.); but the *circle* CHGD cuts CFD in more points than two; viz. in C, G, D; which is impoſſible: wherefore the ſtraight line AB being applied to CD, alſo the ſegment AEB will not, not apply itſelf to CFD; therefore it will apply itſelf, and will be equal to it.

Wherefore the ſimilar ſegments of circles, upon equal ſtraight lines, are equal to one another, Which was to be demonſtrated.

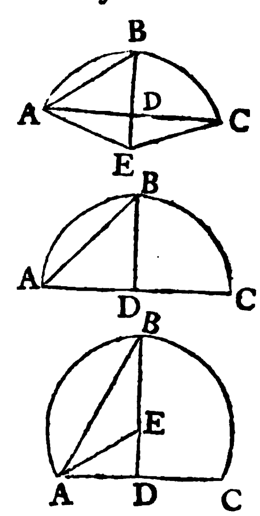
PROP. XXV.

A ſegment of a circle being given to deſcribe the circle of which it is a ſegment.

Let ABC be the given ſegment of the circle; it is required to deſcribe the circle, of which ABC is a ſegment.

For let AC be cut in halves (by 10. 1.) in the *point* D; and from the point D let DB be drawn at right angles to AC; and let AB be joined: therefore the angle ABD is either greater, equal or leſs than the *angle* BAD.

Firſt, let it be greater; and let the *angle* BAE be made, with the ſtraight line AB and at the point A in it, equal to the angle ABD; and let DB be produced to E; and let EC be joined.



Wherefore becauſe the angle ABE is equal to BAE; therefore (by 6. 1.) the ſtraight line BE is equal to EA: And becauſe AD is (by conſt.) equal to DC and DE common; certainly the two AD, DE are equal to the two CD, DE each to each; and the angle ADE is equal to the angle CDE; for each is a right angle; therefore alſo (by 4. 1.) the baſe AE is equal to the baſe EC; but AE has been demonſtrated *to be* equal to EB; therefore alſo BE is equal to CE; wherefore the three *ſtraight lines* AE, EB, EC are equal to one another: therefore a circle deſcribed with

the center E and at the distance of one of the *lines* AE, EB, EC will also pass through the remaining points, and will be the *circle* to be described (by 3. 3.): Wherefore the segment of a circle being given the circle has been described; and it is manifest that the segment ABC is less than a semicircle; on this account, because E the center of it falls without. Book III.

In like manner, if the angle ABD is equal to the angle BAD; AD becoming equal to either of the *lines* BD, DC; therefore the three DA, DB, DC will be equal to one another; and D will be the center of the circle compleated; and certainly ABC will be a semicircle.

But if the *angle* ABD be less than BAD; also let us make with the straight line AB, and at the point A in it, an angle equal to the *angle* ABD; the center will fall within the segment ABC in the *straight line* DB; and certainly the segment ABC will be greater than a semicircle.

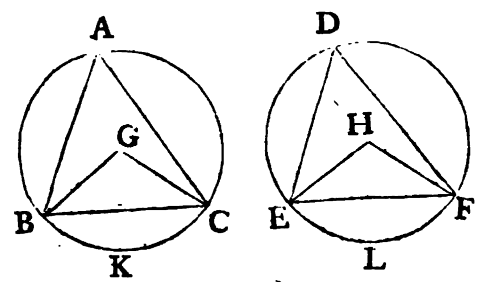
Wherefore a segment of a circle being given, the circle has been described, of which it is a segment. Which was to be done.

PROP. XXVI.

In equal circles, the equal angles stand upon equal circumferences; whether they stand at the center, or at the circumference.

Let ABC, DEF be equal circles; and let BGC, EHF be equal angles in them, at *their* centers; and BAC, EDF at the circumference: I say that the circumference BKC is equal to the circumference ELF.

For let BC, EF be joined.



And because the circles ABC, DEF are equal; the straight lines from their centers are equal (by def. 1. 3.): the two BG, GC are equal to the two EH, HF and (by supp.) the angle at G is equal to the *angle* at H: wherefore the base BC is equal to the base EF: and because the angle at A is equal to the *angle* at D; therefore the segment BAC is

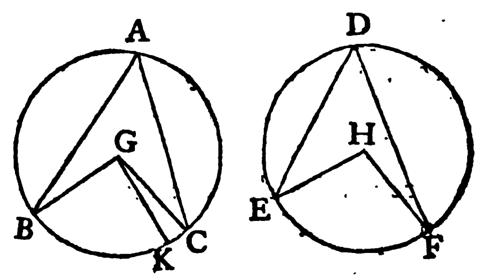
Book III. is similar to the segment EDF (by def. 11. 3.); and they are upon equal straight lines BC, EF; but similar segments *being* upon equal straight lines are equal (by 24. 3.); wherefore the segment BAC *is* equal to the segment EDF; but also the whole circle ABC is equal to the whole circle DEF; therefore the remaining segment BKC is equal to the remaining *segment* ELF; wherefore the circumference BKC is equal to the circumference ELF.

Wherefore in equal circles, the equal angles stand upon equal circumferences; whether they stand at the center or at the circumference. Which was to be demonstrated.

PROP. XXVII.

In equal circles, the angles which stand upon equal circumferences are equal to one another; whether they stand at the center or at the circumference.

For let the angles BGC, EHF stand at the centers G, H in the equal circles ABC, DEF; and upon the equal circumferences BC, EF; and the angles BAC, EDF at the circumferences; I say that the angle *BGC* is equal to the *angle* EHF; and BAC to EDF.



If the *angle* BGC be equal to EHF; it is plain (by 20. 3.) that the *angle* BAC is equal to the *angle* EDF; but if *the angle BGC be* not *equal to the angle* EHF; one of them is greater; let BGC be the greater, and let the *angle* BGK (by 23. 1.) be made with the straight line BG and at the point G in it, equal to the angle EHF: but (by 26. 3.) equal angles stand upon equal circumferences, when they are at the centers; wherefore the circumference BK *is* equal to the circumference EF; but EF (by supp.) is equal to BC; wherefore BK is equal to BC; the less to the greater; which is impossible: wherefore the angle BGC is not unequal to EHF; therefore equal; and the *angle* at A is half of the *angle* BGC; and the angle at D is half of the angle EHF: wherefore the angle at A is equal to the *angle* at D. Where-

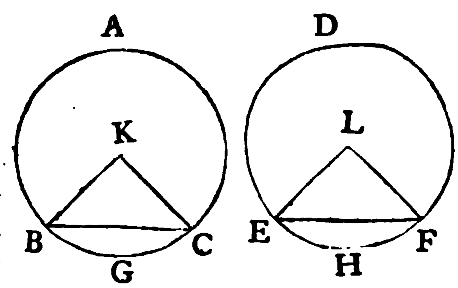
Wherefore in equal circles, the angles which ſtand upon equal circumferences are equal to one another; whether they ſtand at the center or at the circumference. Which was to be demonſtrated. Book III.

PROP. XXVIII.

Equal ſtraight lines in equal circles cut off equal circumferences; the greater *equal* to the greater; and the leſs *equal* to the leſs.

Let ABC, DEF be equal circles; and let BC, EF be equal ſtraight lines in them, cutting off the greater circumferences BAC, EDF; and BGC, EHF the leſs; I ſay that the greater circumference BAC is equal to the greater circumference EDF; and the leſſer circumference BGC *is* equal to the leſs EHF.

For let K, L the centers of the circles be taken; and let BK, KC, EL, LF be joined.



And becauſe the circles are equal; the lines alſo from their centers are eqnal (by def. 1.); therefore the two BK, KC are equal to the two EL, LF; and the baſe BC is (by ſupp.) equal to the baſe EF; wherefore (by 8. 1.) the angle BKC is equal to the angle ELF; but equal angles ſtand upon equal circumferences, when they are at the center (by 26. 3.); therefore the circumference BGC is equal to the circumference EHF; but the whole circle ABC is equal to the whole circle DEF; wherefore the remaining circumference BAC, is equal to the remaining circumference EDF.

Wherefore equal ſtraight lines in equal circles cut off equal circumferences; the greater *equal* to the greater; and the leſs *equal* to the leſs. Which was to be demonſtrated.

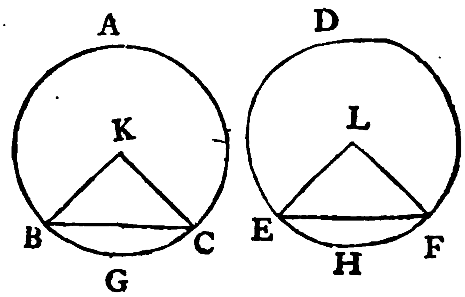
PROP. XXIX.

In equal circles, equal ſtraight lines are extended under equal circumferences.

 Let

Book III. Let ABC, DEF be equal circles; and in them let the equal circumferences BGC, EHF be taken; and let the ſtraight lines BC, EF be drawn; I ſay that the ſtraight line BC is equal to EF.

For let K, L, the centers of the circles be taken; and let BK, KC, EL, LF be joined.



And becauſe the circumference BGC is equal to the circumference EHF; alſo (by 27. 3.) the angle BKC is equal to the *angle* ELF; and becauſe the circles ABC, DEF are equal; the lines from their centers are equal (by def. 1. 3.); therefore the two BK, KC are equal to the two EL, LF; and they contain equal angles; therefore (by 4. 1.) the baſe BC is equal to the baſe EF.

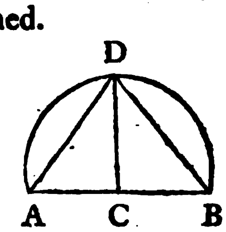
Wherefore, in equal circles, equal ſtraight lines are extended under equal circumferences. Which was to be demonſtrated.

PROP. XXX.

To cut a given circumference in halves.

Let ABD be the given circumference; it is required to cut the given circumference ADB in halves.

Let AB be joined, and let it be cut in halves (by 10. 1.) at the *point* C; and from the point C, let CD be drawn at right angles to the ſtraight line AB; and let AD, DB be joined.



And ſince AC is equal to CB; and CD common; certainly the two AC, CD are equal to the two BC, CD; and the angle ACD is equal to the angle BCD; for each *of theſe is* a right angle; and (by 4. 1.) the baſe AD is equal to the baſe DB; but equal ſtraight lines cut off equal circumferences (by 28. 3.); the greater *equal* to the greater; and the leſs *equal* to the leſs: and each of the circumferences AD, DB is leſs than a ſemicircle; wherefore the circumference AD is equal to the circumference DB.

Where-

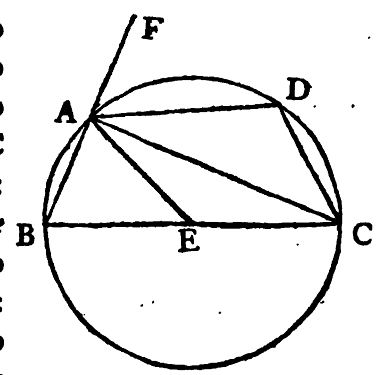
Wherefore the given circumference hath been cut in halves. Which was to be done.

PROP. XXXI.

In a circle, the angle in a semicircle is a right angle; but the *angle* in a greater segment is less than a right angle; and the *angle* in a segment less *than a semicircle is* greater than a right angle: and besides the angle of a greater segment is greater than a right angle; but the angle of a segment less *than a semicircle is* less than a right angle.

Let ABCD be a circle; and let BC be the diameter of it, and E the center; also let BA, AC, AD, DC be joined; I say that the angle in the semicircle BAC is a right angle; but that the angle in the segment ABC greater than a semicircle, viz. the *angle* ABC, *is* less than a right angle; and that the angle in the segment ADC less than a semicircle is greater than a right angle.

Let AE be joined; and let BA be produced to F.



And because BE is equal to EA, the angle EAB (by 5. 1.) is also equal to EBA: again because EA is equal to EC, the angle EAC is also equal to ACE; wherefore the whole *angle* BAC is equal to the two *angles* ABC, ACB: but the *angle* FAC, the outward *angle* of the triangle ABC, is also equal to the two angles ABC, ACB (by 32. 1.): wherefore the angle BAC is equal to the *angle* FAC; therefore *each of them is* a right angle: wherefore the angle BAC, in the semicircle BAC is a right angle.

And since the two angles ABC, BAC of the triangle ABC are less than two right angles; and BAC is a right angle: therefore the angle ABC is less than a right angle; and it is in the segment ABC greater than a semicircle.

And becaese ABCD is a quadrilateral *figure inscribed* in a circle; and the opposite angles of quadrilateral *figures* in circles are (by

22. 3.)

Book III. 22. 3.) equal to two right angles; wherefore the angles ABC, ADC are equal to two right angles; but ABC is leſs than a right angle; therefore the remainder ADC is greater than a right angle; and it is in a ſegment leſs than a ſemicircle.

I ſay alſo that the angle of a greater ſegment, the angle contained by the circumference ABC and the ſtraight line AC, is greater than a right angle; but the angle of the ſegment leſs *than a ſemicircle*, the *angle* contained by the circumference ADC and the ſtraight line AC is leſs than a right angle: and it is manifeſt from hence. For becauſe the angle contained by the ſtraight lines BA, AC is a right angle; therefore the *angle* contained by the circumference ABC and the ſtraight line AC is greater than a right angle (by com. not. 9.): Again becauſe the *angle contained* by the ſtraight lines CA, AF is a right angle; therefore the *angle contained* by the ſtraight line AC and the circumference ADC is leſs than a right angle.

OTHERWISE. A demonſtration that BAC is a right angle. Becauſe the *angle* AEC (by 32. 1.) is double of the *angle* BAE; for it is equal to the two inward and oppoſite *angles*; and AEB is alſo double of the *angle* EAC; wherefore the *angles* AEB, AEC are double of the angle BAC; but the *angles* AEB, AEC are (by 13. 1.) equal to two right angles; wherefore the *angle* BAC is a right angle. Which was to be demonſtrated.

Cor From this it is manifeſt that if one angle of a triangle be equal to *the other* two it is a right angle: for this *reaſon* becauſe the adjacent angle is equal to the ſame two: but (by def. 10. 1.) when the adjacent angles are equal, they are right angles.

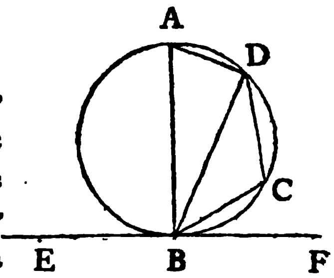
PROP. XXXII.

If any ſtraight line touch a circle; and *if* from the contact any ſtraight line be drawn to the circle, cutting the circle; the angles which it makes with the touching *line* will be equal to the angles in the alternate ſegments of the circle.

For let any ſtraight line EF touch the circle ABCD in the point B; and from the point B let BD any ſtraight line be drawn to the circle ABCD cutting it; I ſay that the angles which BD makes with

with the touching line EF will be equal to the angles in the alternate ſegments of the circle, that is, that the angle FBD is equal to the angle conſtituted in the ſegment DAB; and the angle EBD is equal to the *angle* in the ſegment DCB. Book III.

For let BA be drawn from the *point* B at right angles to EF; and let C any accidental point be taken in the circumference BD; and let AD, DC, CB be joined.



And becauſe a certain ſtraight line EF touches the circle ABCD in the point B; and from the contact at B the *ſtraight line* AB has been drawn at right angles to the touching *line*; the center of the circle ABCD is (by 19. 3.) in the *ſtraight line* AB: therefore the angle ADB, being in a ſemicircle, is (by 31. 3.) a right angle; therefore BAD, ABD, the remaining *angles*, are equal to one right angle: but ABF is alſo a right angle (by conſt.); wherefore the angle ABF is equal to the angles BAD, ABD; let the common angle ABD be taken away; therefore the remaining angle DBF is equal to the angle in the alternate ſegment of the circle, viz. the *angle* BAD: And becauſe ABCD is a quadrilateral *figure inſcribed* in a circle, its oppoſite angles are (by 22. 3.) equal to two right angles; therefore the *angles* DBF, DBE are equal to BAD, BCD; of which BAD has been demonſtrated to be equal to DBF; therefore the remaining *angle* DBE is equal to the *angle* in the alternate ſegment DCB of the circle, viz. the angle DCB.

Wherefore if any ſtraight line touch a circle; and if from the contact any ſtraight line be drawn to the circle, cutting the circle; the angles which it makes with the touching *line* will be equal to the angles in the alternate ſegments of the circle. Which was to be demonſtrated.

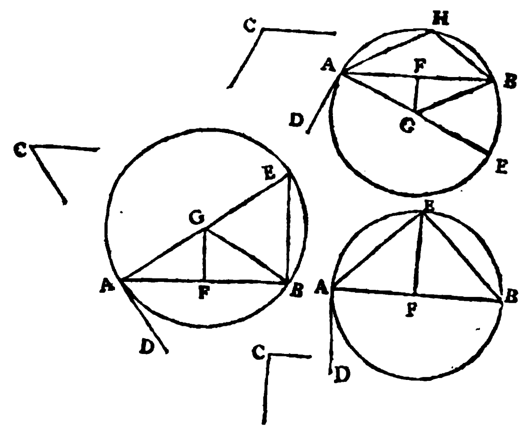
PROP. XXXIII.

Upon a given ſtraight line to deſcribe a ſegment of a circle, containing an angle equal to a given rectilineal angle.

Let the given ſtraight line be AB; and the *angle* at C, the given rectilineal angle: it is required, upon the given ſtraight line AB, to

Book III. to defcribe a fegment of a circle containing an angle equal to the *angle* at C: but the *angle* at C is either an acute *angle* or a right angle or an obtufe *angle*.

Firft let it be an acute *angle*, as in the firft figure; and let the *angle* BAD be made, with the ftraight line AB and at the point A in it, equal to the angle at C; wherefore alfo the angle BAD is acute: and let AE be drawn, from the point A, at right angles to AD; and let AB be cut in halves at F; and from the point F let FG be drawn at right angles to AB; and let GB be joined.



And becaufe AF is equal to FB and FG common; certainly the two AF, FG are equal to the two BF, FG; and the angle AFG is (by conft.) equal to BFG; therefore (by 4. 1.) the bafe AG is equal to the bafe GB; therefore the circle defcribed with the center G and at the diftance AG, will alfo pafs through B: let it be defcribed; and let it be ABE; and let BE be joined. Wherefore becaufe from the extremity of the diameter AE; from the *point* A; AD is at right angles to AE; therefore AD touches the circle (by cor. to 16. 3.). And fince a certain ftraight line AD touches the circle ABE; and from the contact at A a certain ftraight line AB hath been drawn to the circle ABE; therefore the angle DAB is equal to the angle AEB in the alternate fegment of the circle; but the angle DAB is equal to the angle at C; therefore the angle at C is equal to the angle AEB: wherefore upon the given ftraight line AB, a fegment of a circle hath been defcribed viz. AEB, containing the angle AEB equal to the given angle at C.

But let the *angle* at C be a right angle; and again let it be required to defcribe upon AB a fegment of a circle containing an angle equal to the right angle at C: Again let the *angle* BAD be made equal to the right angle at C, as it is in the fecond figure; and let AB be cut in halves in F; and with the center F, and at the

the distance of either of *the lines* AF, FB let the circle AEB be described. Therefore the straight line AD touches the circle ABE; because the angle at A is a right angle; and the angle BAD is equal to the *angle* in the segment AEB; for it is also a right angle (by 31. 3.) being in a semicircle: but the *angle* BAD is equal to the *angle* at C: wherefore, again a segment of a circle viz. AEB has been described upon AB, containing an angle equal to the *angle* at C.

But let the *angle* at C be an obtuse *angle*: and let the angle BAD be made equal to it, with the straight line AB and at the point A, as it is in the third figure; and let AE be drawn at right angles to AD; and again let AB be cut in halves at F; and let FG be drawn at right angles to AB; and let GB be joined.

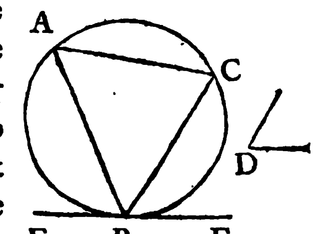
And again, because AF is equal to FB and FG common; certainly the two AF, FG are equal to the two BF, FG; and the angle AFG is equal to the angle BFG: therefore the base AG is equal to the base GB; wherefore a circle described with the center G; and at the distance AG, will also pass through B; let it pass as AEB: And because AD has been drawn at right angles to the diameter AE from *its* extremity: therefore AD (by cor. 16. 3.) touches the circle AEB: and from the contact at A, the *straight line* AB hath been drawn; wherefore the angle BAD is equal to AHB the angle contained in the alternate segment of the circle: But the angle BAD is equal to the *angle* at C: therefore also the angle in the segment AHB is equal to the *angle* at C. Wherefore upon the given straight line AB, a segment of a circle; viz. AHB, has been described, containing an angle equal to the *angle* at C. Which was to be done.

PROP. XXXIV.

To cut off a segment from a given circle, containing an angle equal to a given rectilineal angle.

Let ABC be the given circle; and the given rectilineal angle, the *angle* at D: it is required to cut off a segment from the circle ABC, containing an angle equal to the angle at D.

Book III. Let EF be drawn touching the circle ABC in the point B; and let the angle FBC be made with the straight line EF, and at the point B in it, equal to the angle at D.



Wherefore becauſe a certain ſtraight line EF touches the circle ABC, and from the contact at B, BC hath been drawn *cutting it*; therefore (by 32. 3.) FBC is equal to the angle contained in the alternate ſegment BAC: but the *angle* FBC is equal to the angle at D; wherefore alſo the angle in the ſegment BAC is equal to the angle at D.

Wherefore from a given circle ABC a ſegment BAC has been cut off, containing an angle equal to the given rectilineal angle at D. Which was to be done.

PROP. XXXV.

If in a circle two ſtraight lines cut one another, the rectangle contained by the ſegments of the one is equal to the rectangle contained by the ſegments of the other.

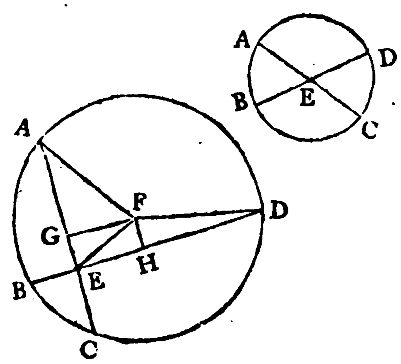
For in the circle ABCD let the two ſtraight lines AC, BD cut one another in the point E; I ſay that the rectangle contained by AE, EC is equal to the rectangle contained by DE, EB.

If AC, BD paſs through the center ſo that E be the center of the circle ABCD: it is manifeſt, AE, EC, DE, EB being equal, that the rectangle contained by AE, EC is equal to the rectangle contained by DE, EB.

But let AC, DB not paſs through the center; and let the center of the circle ABCD be taken (by 1. 3.); and let it be F; and from the *point* F, let FG, FH be drawn perpendiculars to AC, DB; and let FD, FA, FE be joined.

And becauſe GF a certain ſtraight line through the center cuts a certain ſtraight line AC not through the center at right angles; it will alſo cut it in halves (by 3. 3.); wherefore AG is equal to GC: wherefore becauſe the ſtraight line AC has been cut into equal *ſegments* at the *point* G; and into unequal *ſegments* at the *point* E; therefore (by 5. 2.) the rectangle contained by AE and

EC



Book III.

EC together with the ſquare of GE is equal to the ſquare of GA; let the *ſquare* of GF *which is* common be added: wherefore the *rectangle contained* by AE, EC together with the *ſquares* of GE, GF is equal to the *ſquares* of AG, GF: But the *ſquare* of FE is equal to the *ſquares* of EG, GF (by 47. 1.); and the *ſquare* of FA is equal to the *ſquares* of AG, GF: wherefore the rectangle contained by AE, EC together with the *ſquare* of FE is equal to the ſquare of FA: but FA is equal to FD; wherefore the *rectangle contained* by AE, EC together with the *ſquare* of FE is equal to the *ſquare* of FD. Certainly for the ſame reaſon alſo the *rectangle contained* by DE, EB together with the *ſquare* of FE is equal to the ſquare of FD: but it has alſo been demonſtrated that the *rectangle contained* by AE, EC together with the ſquare of FE is equal to the ſquare of FD: wherefore the rectangle contained by AE, EC together with the ſquare of FE is equal to the rectangle contained by DE, EB together with the ſquare of FE: let the ſquare of FE *which is* common be taken away; therefore the remaining rectangle contained by AE, EC is equal to the *remaining* rectangle contained by DE, EB.

Wherefore if in a circle two ſtraight lines cut one another, the rectangle contained by the ſegments of the one is equal to the rectangle contained by the ſegments of the other. Which was to be demonſtrated.

PROP. XXXVI.

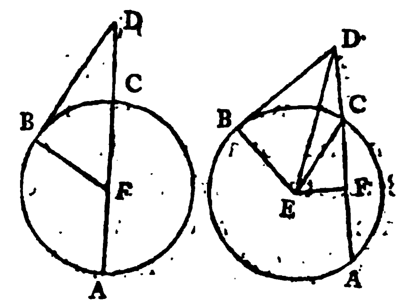
If any point be taken without a circle, and two ſtraight lines fall from it upon the circle; and one of them cuts the circle and the other touches it; the rectangle contained by the whole cutting line and the ſegment without, taken between the point and the convex circumference, will be equal to the ſquare of the touching *line*.

For let D, any point without the circle ABC, be taken; and from the *point* D, let the two ſtraight lines DCA, DB fall upon

 the

Book III. the circle ABC; and let DCA cut the circle ABC; and let DB touch *it*: I say that the rectangle contained by AD, DC is equal to the square of DB: the *straight line* DCA either passes through the center or not.

First let it pass through the center; and let F be the center of the circle ABC; and let FB be joined; therefore (by 18. 3.) FBD is a right angle: And because the straight line AC hath been cut in halves in the *point* F, and CD is added to it; therefore (by 6. 2.) the *rectangle contained* by AD, DC together with the *square* of FC is equal to the *square* of FD: but the *square* of FD (by 47. 1.) is equal to the squares of FB, BD; for the angle FBD is a right angle: wherefore the rectangle contained by AD, DC together with the *square* of FB is equal to the *squares* of FB, BD; let the common *square* of FB be taken away; therefore the remaining *rectangle contained* by AD, DC is equal to the *remaining square* of DB the touching *line*.



But let DA not pass through the center of the circle ABC: and let E the center of the circle be taken; and from E let EF be drawn perpendicular to AC; and let EB, EC, ED be joined; wherefore EFD is a right angle; and because EF a certain straight line through the center, cuts at right angles AC a certain straight line not through the center, it will also (by 3. 3.) cut it in halves; therefore AF is equal to FC: and because the straight line AC hath been cut in halves at F; and CD is added to it; therefore (by 6. 2.) the *rectangle contained* by AD, DC together with the *square* of FC is equal to the *square* of FD: let the common *square* of FE be added; therefore the rectangle contained by AD, DC together with the squares of CF, FE are equal to the squares of DF, FE: but the square of DE is equal to the squares of DF, FE (by 47. 1.); for EFD is a right angle; and the square of CE is equal to the squares of CF, FE: Wherefore the *rectangle contained* by AD, DC together with the square of EC is equal to the square of ED: but CE is equal to EB; therefore the *rectangle contained* by AD, DC together with the square

square of EB is equal to the square of ED: but the square of ED is equal to the squares of EB, BD; for the angle EBD is a right angle: wherefore the rectangle contained by AD, DC together with the square of EB is equal to the squares of EB, BD; let the common square of EB be taken away; therefore the remaining *rectangle contained* by AD, DC is equal to the *remaining square* of DB.

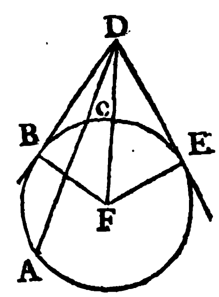
Wherefore if any point be taken without a circle, &c. Which was to be demonstrated.

PROP. XXXVII.

If any point be taken without a circle, and two straight lines fall from the point upon the circle, and one of them cuts the circle and the other meets *it*, and *if* the *rectangle contained* by the whole cutting line, and the *segment* without, taken between the point and the convex circumference, be equal to the *square* of the line which meets it; the meeting *line* will touch the circle.

For let D be taken any point without the circle ABC; and let the two straight lines DCA, DB fall from the point D, upon the circle ABC; and let DCA cut the circle, and DB meet *it*; and let the *rectangle contained* by AD, DC be equal to the *square* of DB; I say that DB touches the circle ABC.

For let DE be drawn (by 17. 3.) touching the circle ABC; and let F the center of the circle ABC be taken (by 1. 3.); and let FE, FB, FD be joined: therefore the *angle* FED is (by 18. 3.) a right angle.



And since DE touches the circle ABC; and DCA cuts *it*; therefore (by 36. 3.) the *rectangle contained* by AD, DC is equal to the *square* of DE; but the *rectangle contained* by AD, DC is supposed equal to the *square* of DB; therefore the square of DE is equal to the square of DB; wherefore DE is equal to DB; and FE is also equal to FB; therefore the two DE, EF are equal to the two DB, BF; and DF is a common base to them; therefore (by 8. 1.) the angle DEF is equal to the angle DBF; but DEF is a right

 a right angle; therefore DBF is alſo a right angle; and FB produced is a diameter; but the *ſtraight line* drawn from the extremity at right angles to the diameter touches the circle ABC (by cor. to 16. 3.); certainly it will be demonſtrated in the ſame manner if the center happen to be in AC.

Wherefore if any point be taken without a circle, &c. Which was to be demonſtrated.

THE

ELEMENTS

OF

EUCLID.

BOOK IV.

DEFINITIONS.

Book IV.

1. A Rectilineal figure is ſaid to be inſcribed in a rectilineal figure, when each of the angles of the inſcribed figure touches each ſide of the figure in which it is inſcribed. 2. And in like manner a figure is ſaid to be circumſcribed about a figure, when each ſide of the circumſcribed *figure* touches each angle of the figure about which it is circumſcribed. 3. And a rectilineal figure is ſaid to be inſcribed in a circle when each angle of the inſcribed *figure* touches the circumference of the circle. 4. And a rectilineal figure is ſaid to be circumſcribed about a circle, when each ſide of the circumſcribed figure (*is a tangent to the circle*) touches the circumference of the circle. 5. In like manner a circle is ſaid to be inſcribed in a figure, when the circumference of the circle touches each ſide of the *figure* within which it is inſcribed. 6. But a circle is ſaid to be circumſcribed about a figure, when the cir-

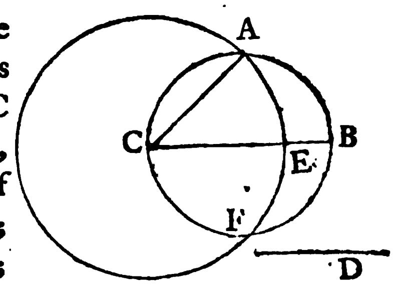
Book IV. circumference of the circle touches each angle of the *figure* about which it is circumfcribed.

7. A ftraight line is faid to be applied *or placed* in a circle, when the extremities of it are in the circumference of the circle.

PROP. I.

In a given circle to apply a ftraight line, equal to a given ftraight line, which is not greater than the diameter.

Let the given circle be ABC; and D the given ftraight line, not greater than the diameter of the circle: it is required to apply in the circle ABC a ftraight line equal to the ftraight line D.



Let BC the diameter of the circle ABC be drawn: if therefore BC be equal to D, the thing required has been done: for the ftraight line BC has been applied, in the circle ABC, equal to the ftraight line D. But if not, BC is greater than D (by fupp.); and (by 3. 1.) make CE equal to D; and with the center C, and at the diftance CE let the circle AEF be defcribed; and let CA be joined.

Wherefore becaufe the point C is the center of the circle AEF CA is equal to CE; but D is equal to CE; therefore alfo D is equal to CA.

Wherefore in the given circle ABC, the ftraight line AC has been applied, equal to the given ftraight line D, which is not greater than the diameter of the circle. Which was to be done.

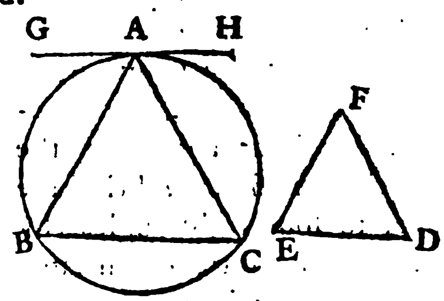
PROP. II.

In a given circle, to infcribe a triangle equiangular to a given triangle.

Let ABC be the given circle, and DEF the given triangle; it is required to infcribe in the circle ABC a triangle equiangular to the triangle DEF.

Let

Let GAH be drawn, touching the circle ABC in the *point* A; Book IV. and let the angle HAC be made with the ſtraight line AH, and at the *point* A in it, equal to the angle DEF: and the *angle* GAB, with the ſtraight line AG, and at the *point* A in it, equal to the angle FDE; and let BC be joined.



Therefore becauſe a certain ſtraight line HAG touches the circle ABC, and a certain *ſtraight line* AC hath been drawn from the contact *cutting it*; therefore (by 32. 3.) the angle HAC is equal to the *angle* ABC, the angle in the alternate ſegment of the circle: but the angle HAC is equal to the angle DEF; therefore the angle ABC is equal to the *angle* FED. Certainly for the ſame reaſon alſo the angle ACB is equal to FDE; and therefore (by 32. 1.) the remaining angle BAC is equal to the remaining angle EFD: wherefore the triangle ABC is equiangular to the triangle DEF; and it has been inſcribed in the circle ABC (by def. 3. 4.).

Wherefore a triangle equiangular to the given triangle hath been inſcribed in the given circle. Which was to be done.

PROP. III.

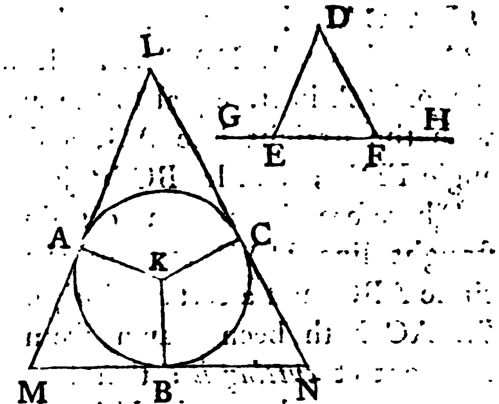
To circumſcribe a triangle, about a given circle, equiangular to a given triangle.

Let ABC be the given circle, and DEF the given triangle; it is required to circumſcribe a triangle about the circle ABC equiangular to the triangle DEF.

Let EF be produced towards both parts, to the points H, G; and let K the center of the circle ABC be taken; and let the ſtraight line KB be drawn as it may happen: and let the angle BKA be made with the ſtraight line KB and at the point K in it equal to the angle DEG; and BKC equal to the angle DFH; and through the points A, B, C let the *ſtraight lines* LAM, MBN, NCL be drawn touching the circle ABC.

 and

Book IV. And becaufe LM, MN, NL touch the circle ABC in the points, A, B, C; and from the center K, KA, KB, KC have been drawn to the points A, B, C; therefore (by 18. 3.) the angles at the points A, B, C are right angles. And becaufe the four angles of the quadrilateral figure AMBK are equal to four right angles (by 32. 1.); [fince the quadrilateral figure AMBK is divifible into two triangles] of which the angles KAM, KBM are two right angles; wherefore the remaining angles AKB, AMB are equal to two right angles; but the *angles* DEG, DEF are alfo (by 13. 1.) equal to two right angles; wherefore the *angles* AKB, AMB are equal to the *angles* DEG, DEF; of which AKB is (by conft.) equal to DEG; therefore the remaining *angle* AMB is equal to the remaining *angle* DEF. Certainly in the fame manner it will be demonftrated that the *angle* LNM is equal to the *angle* DFE; therefore the remaining *angle* MLN is equal to the remaining *angle* EDF (by 32. 1.); therefore the triangle LMN is equiangular to the triangle DEF and (by def. 4. 4.) it is circumfcribed about the circle ABC.



Wherefore a triangle has been circumfcribed about the given circle, equiangular to the given triangle. Which was to be done.

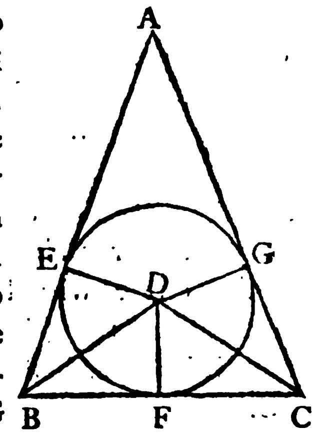
PROP. IV.

To infcribe a circle in a given triangle.

Let ABC be the given triangle; it is required to infcribe a circle in the triangle ABC.

Let the angles ABC, BCA be cut in halves by the ftraight lines BD, CD; and let them meet one another in the point D; and from the *point* D let DE, DF, DG be drawn perpendiculars to the ftraight lines AB, BC, CA.

And becaufe the angle ABD is equal to the angle CBD, for ABC is cut in halves; alfo the right angle BED is equal to the right



right angle BFD; therefore there are two triangles EBD, DBF having two angles equal to two angles, and one ſide equal to one ſide; viz. BD common to both extended under one of the equal angles; alſo (by 26. 1.) they will therefore have the remaining ſides equal to the remaining ſides; therefore DE *is* equal to DF: Certainly for the ſame reaſon alſo DG is equal to DF; wherefore the circle deſcribed with the center D and at the diſtance of *any* one of the *lines* DE, DF, DG will paſs alſo through the remaining points, and will touch the ſtraight lines AB, BC, CA; on account of the angles at the points E, F, G being right angles; for if it ſhall cut them; there will be a *ſtraight line*, drawn at right angles to the diameter from *its* extremity, falling within the circle, which (by 16. 3.) is abſurd: wherefore the circle deſcribed with the center D and at the diſtance of *any* one of the *lines* DE, DF, DG does not cut the ſtraight lines AB, BC, CA; therefore it will touch them; and will be a circle inſcribed in the triangle ABC.

Wherefore the circle EFG is inſcribed in the given triangle ABC. Which was to be done.

PROP. V.

To circumſcribe a circle about a given triangle.

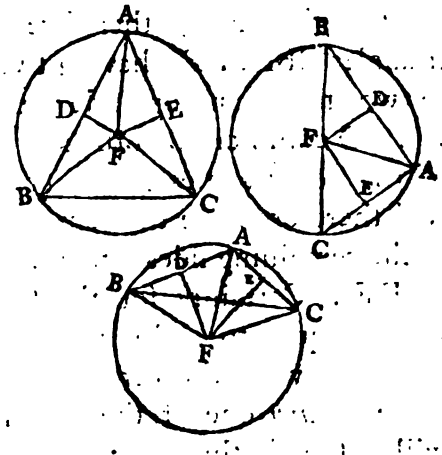
Let ABC be the given triangle: it is required to circumſcribe a circle about the given triangle ABC.

Let AB, AC be cut in halves in the points D and E; and let DF and EF be drawn from the points D, E at right angles to AB, AC; they will meet either within the triangle ABC or in the ſtraight line BC or without the triangle ABC.

Firſt let them meet within at the point F, and let BF, FC, FA be joined: and becauſe AD is equal to DB; and DF common and at right angles; therefore (by 4. 1.) the baſe AF is equal to the baſe FB: Certainly in like manner we ſhall demonſtrate that CF is alſo equal to FA; ſo that (by com. not. 1.) BF *is* alſo equal to

 FC;

Book IV. FC; therefore the three straight lines FA, FB, FC are equal to one another; wherefore a circle described with the center F and at the distance of *any* one of the *lines* FA, FB, FC will also pass through the remaining points; and the circle will be circumscribed about the triangle ABC; and let it be described as the circle ABC.



But let DF, EF meet in the straight line BC, as it is in the second figure; and let AF be joined: Certainly in the same manner we shall demonstrate that the point F is the center of the circle circumscribed about the triangle ABC.

But let DF, EF meet without the triangle ABC, again in the point F; as it is in the third figure; and let AF, FB, FC be joined; and again because AD is equal to DB, and DF common and at right angles; therefore (by 4. 1.) the base AF is equal to the base EB: Certainly in the same manner we shall demonstrate that CF is equal to FA; so that also BF is equal to FC; therefore again the circle described with the center F and at the distance of *any* one of the *lines* FA, FB, FC will also pass through the remaining points; and will be circumscribed about the triangle ABC; and let it be described as ABC.

Wherefore a circle has been circumscribed about the given triangle. Which was to be done.

Cor. And *it is* manifest that, when the center of the circle falls within the triangle, the angle BAC being in a segment greater than a semicircle, is less than a right angle; but when *it falls* in BC being in a semicircle, it will be a right angle; but when the center falls without the triangle ABC, the angle BAC being in a segment less than a semicircle, is greater than a right angle. So that also when the given triangle is acute angled the straight lines DF, EF will meet within the triangle; but when BAC is a right angle; *they will meet* in BC; but when it is greater than a right angle, without the triangle ABC.

PROP.

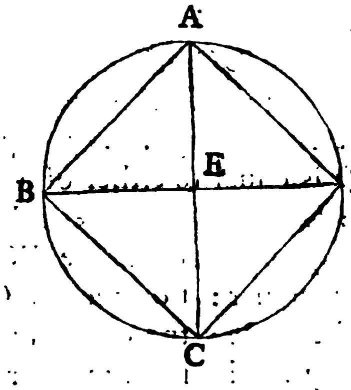
PROP. VI.

To inſcribe a ſquare in a given circle.

Let ABCD be the given circle: it is required to inſcribe a ſquare in the circle ABCD.

Let AC, BD, diameters of the circle ABCD, be drawn at right angles to one another; and let AB, BC, CD, DA be joined.

And becauſe BE is equal to ED; for E is the center; and EA common and at right angles; therefore (by 4. 1.) the baſe AB is equal to the baſe AD: certainly for the ſame reaſon alſo each of the *lines* BC, CD is equal to each of the *lines* BA, AD; therefore the quadrilateral figure ABCD is equilateral. I ſay it is alſo rectangular: for becauſe the ſtraight line BD is a diameter of the circle ABCD; therefore BAD is a ſemicircle; therefore the angle BAD is a right angle (by 31. 3.): Certainly for the ſame reaſon alſo each of the *angles* ABC, BCD, CDA is a right angle; therefore the quadrilateral figure ABCD is rectangular: but it has been demonſtrated to be equilateral; therefore it is a ſquare; and it has been inſcribed in the given circle ABCD.



Wherefore the ſquare ABCD has been inſcribed in the given circle ABCD. Which was to be done.

PROP. VII.

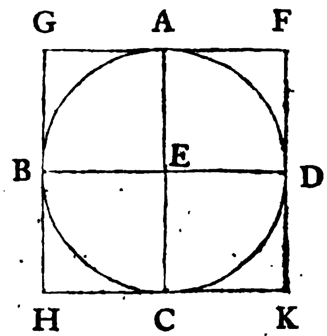
To circumſcribe a ſquare about a given circle.

Let ABCD be the given circle: it is required to circumſcribe a ſquare about the circle ABCD.

Let AC, BD two diameters of the circle ABCD be drawn at right angles to one another; and through the points A, B, C, D let FG, GH, HK, KF be drawn touching the circle ABCD.

Wherefore ſince FG touches the circle ABCD; and from the center E to the contact at A, the *ſtraight line* AE hath been drawn; therefore the angles at A are right angles: Certainly for the ſame reaſon

Book IV. reaſon the angles at the points B, C, D are right angles; and becauſe the angle AEB is a right angle (by conſt.); as alſo the angle EBG (by cor. to 16. 3.) is a right angle; therefore (by 29. 1.) GH is parallel to AC: Certainly for the ſame reaſon alſo AC is parallel to FK: and in the ſame manner we ſhall demonſtrate that each of the *lines* GF, HK is parallel to BED: wherefore GK, GC, AK, FB, BK are parallelograms; therefore (by 34. 1.) GF is equal to HK; and GH to FK; and becauſe AC is equal to BD; but AC (by 34. 1.) is equal to either of the *lines* GH, FK; and BD is equal to either of the *lines* GF, HK; therefore alſo each of the *lines* GH, FK is equal to each of the *lines* GF, HK; therefore the quadrilateral figure FGHK is equilateral: I ſay that it is alſo rectangular: For becauſe GBEA is a parallelogram; and AEB is a right angle; therefore (by 34. 1.) AGB *is* alſo a right angle; certainly in the ſame manner we ſhall demonſtrate, that the angles at the *points* H, K, F are right angles; therefore the quadrilateral figure FGHK is rectangular; and it has been alſo demonſtrated to be equilateral; therefore it is a ſquare and has been circumſcribed about the circle ABCD.



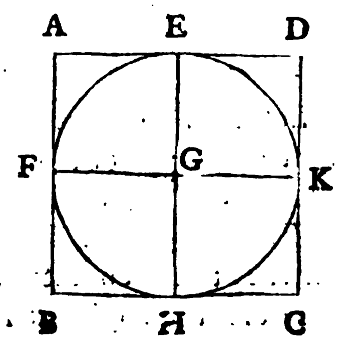
Wherefore a ſquare has been circumſcribed about the given circle. Which was to be done.

PROP. VIII.

To inſcribe a circle in a given ſquare.

Let ABCD be the given ſquare: it is required to inſcribe a circle in the ſquare ABCD.

Let each of the *lines* AB, AD be cut in halves at the points F, E; and through the point E, let EH be drawn parallel to either of the *lines* AB, CD; and through the point F, let FK be drawn parallel to either of the *lines* AD, BC; therefore each of the *figures* AK, KB; AH, HD; AG, GC; BG, GD is a parallelogram; and certainly (by 34. 1.) the oppoſite ſides of



them are equal: and because AD is equal to AB (by supp.); and AE is the half of AD; and AF the half of AB; therefore AE is equal to AF; so that also the opposite *sides* are equal; therefore FG is equal to GE: Certainly in the same manner we shall demonstrate that each of the *lines* GH, GK is equal to each of the lines FG, GE; therefore the four GE, GF, GH, GK are equal to one another; therefore the circle described with G for a center and at the distance of *any* one of the *lines* GE, GF, GH, GK will also pass through the remaining points; and will touch the straight lines AB, BC, CD, DA; on account of the angles at the *points* E, F, H, K being right angles (by cor. to 16. 3.); for if the circle will cut the *straight lines* AB, BC, CD, DA; the straight line drawn at right angles to the diameter of the circle from its extremity will fall within the circle; which is absurd (by 16. 3.); wherefore the circle described with the center G and at the distance of *any* one of the *lines* GE, GF, GH, GK does not cut the straight lines AB, BC, CD, DA; therefore it will touch them and will be inscribed in the square ABCD. Book IV.

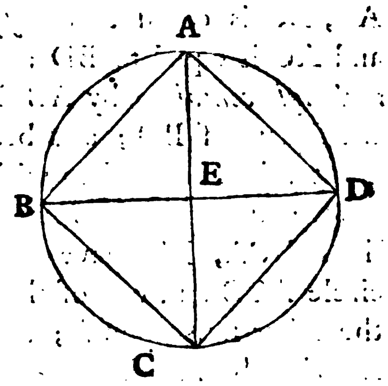
Wherefore a circle has been inscribed in a given square. Which was to be done.

PROP. IX.

To circumscribe a circle about a given square.

Let ABCD be the given square: it is required to circumscribe a circle about the square ABCD.

For AC and BD being joined let them cut one another in the point E.



And because DA is equal to AB; and AC common; certainly the two DA, AC are equal to the two BA, AC and the base DC is equal to the base BC; therefore (by 8. 1.) the angle DAC is equal to the angle BAC; therefore the angle DAB has been cut in halves by AC: Certainly in the same manner we shall demonstrate that each of the angles ABC, BCD, CDA hath been cut in

halves

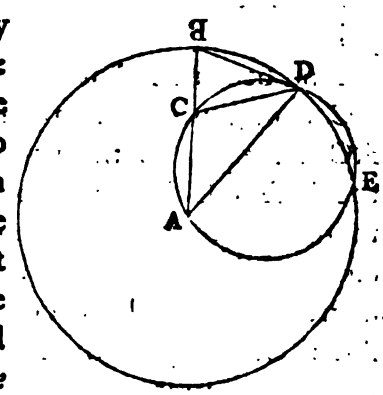
Book IV. halves by the ſtraight lines AC, DB: and becauſe the angle DAB is equal to ABC; and EAB is the half of DAB; and EBA *is* the half of ABC; therefore the *angle* EAB is equal to the *angle* EBA; ſo that alſo (by 6. 1.) the ſide EA is equal to the ſide EB: Certainly in the ſame manner we ſhall demonſtrate that each of the ſtraight lines EC, ED is equal to each of the ſtraight lines EA, EB; therefore the four *ſtraight lines* EA, EB, EC, ED are equal to one another; wherefore a circle deſcribed with the center E and at the diſtance of *any* one of the lines EA, EB, EC, ED will paſs alſo through the remaining points; and will be circumſcribed about the ſquare ABCD.

Wherefore a circle has been circumſcribed about a given ſquare. Which was to be done.

PROP. X.

To make an iſoſceles triangle, having each of the angles at the baſe double of the remaining *angle*.

Draw any ſtraight line AB; and let it be cut in the point C, (by 11. 2.) ſo that the rectangle contained by AB, BC may be equal to the ſquare of CA; and with the center A and at the diſtance AB let the circle BDE be deſcribed: and let BD a ſtraight line equal to the ſtraight line AC, which is not greater than the diameter of the circle BDE be applied in the circle BDE (by 1. 4.): and let DA, DC be joined; and let the circle ACD be circumſcribed (by 5. 4.) about the triangle ACD.



And becauſe the *rectangle contained* by AB, BC is equal to the *ſquare* of AC: and AC is equal to BD; wherefore the *rectangle contained* by AB, BC is equal to the ſquare of BD; and becauſe a certain point B has been taken without the circle ACD; and from the *point* B, two ſtraight lines BCA, BD have fallen upon the circle ACD and one of them cuts *it*, and the other falls upon *it*; and the *rectangle contained* by AB, BC is equal to the *ſquare*

of

of BD; wherefore (by 37. 3.) BD touches the circle ACD; wherefore becauſe BD touches *it*, and from the contact at D: DC has been drawn *cutting it*; therefore (by 32. 3.) the angle BDC is equal to the angle DAC in the alternate ſegment of the circle: wherefore becauſe the *angle* BDC is equal to DAC let the *angle* CDA which is common be added; therefore the whole *angle* BDA is equal to the two *angles* CDA, DAC: But (by 32. 1.) the outward *angle* BCD is equal to the *angles* CDA, DAC; therefore the *angle* BDA is equal to the *angle* BCD: But the *angle* BDA is equal to CBD (by 5. 1.); ſince the ſide AB is equal to AD; ſo that alſo the *angle* DBA is equal to the *angle* BCD: therefore the three *angles* BDA, DBA, BCD are equal to one another: and ſince the angle DBC is equal to BCD (by 6. 1.) the ſide BD is alſo equal to the ſide DC; but BD is put equal to CA; therefore alſo AC is equal to CD; ſo that the angle CDA is equal to DAC; therefore the angles CDA, DAC are the double of DAC; but the *angle* BCD is equal to the *angles* CDA, DAC; therefore BCD is alſo the double of DAC; but BCD is equal to either of the *angles* BDA, DBA; therefore each of the *angles* BDA, DBA is double of the *angle* DAB. Book IV.

Therefore an iſoſceles triangle ADB has been made, having each of the angles at the baſe BD double of the remaining *angle*. Which was to be done.

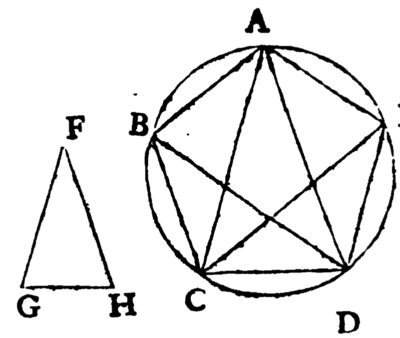
PROP. XI.

To inſcribe an equilateral and equiangular pentagon in a given circle.

Let ABCDE be the given circle; it is required to inſcribe an equilateral and equiangular pentagon in the circle ABCDE.

Make (by 10. 4.) the iſoſceles triangle FGH having each of the angles at G and H double of the angle at F: and inſcribe (by 2. 4.) the triangle ACD in the circle ABCDE equiangular to the triangle FGH; ſo that the *angle* CAD may be equal to the angle at F; and each of the *angles* at G and H equal to each of the angles ACD, CDA; therefore alſo each of the *angles* ACD, CDA is double of the *angle* CAD: Let each of the *angles* ACD, CDA be cut in halves (by 9. 1.) by the ſtraight lines CE, DB; and let AB, BC, DE, EA be drawn.

Book IV.



Wherefore becauſe each of the angles ACD, CDA is double of the angle CAD; and they have been cut in halves by the ſtraight lines CE, DB; therefore the five angles DAC, ACE, ECD, CDB, BDA are equal to one another; but (by 26. 3.) equal angles ſtand upon equal circumferences; therefore the five circumferences AB, BC, CD, DE, EA are equal to one another: but (by 29. 3.) equal ſtraight lines are extended under equal circumferences; therefore the five ſtraight lines AB, BC, CD, DE, EA are equal to one another; therefore the pentagon ABCDE is equilateral; I ſay it is alſo equiangular; for becauſe the circumference AB is equal to the circumference DE; let the common *circumference* BCD be added; therefore the whole circumference ABCD is equal to the whole circumference EDCB: And the angle AED ſtands upon the circumference ABCD; and the angle BAE *ſtands* upon the circumference EDCB; and therefore (by 27. 3.) the angle BAE is equal to the angle AED: Certainly for the ſame reaſon alſo each of the angles ABC, BCD, CDE are equal to either of the angles BAE, AED: therefore the pentagon ABCDE is equiangular; but it has alſo been demonſtrated to be equilateral.

Wherefore an equilateral and equiangular pentagon has been inſcribed in a given circle. Which was to be done.

PROP. XII.

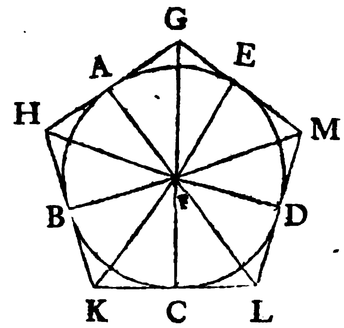
To circumſcribe an equilateral and equiangular pentagon about a given circle.

Let ABCDE be the given circle; it is required to circumſcribe an equilateral and equiangular pentagon about the circle ABCDE.

Let A, B, C, D, E be underſtood to be the points of the angles of a pentagon inſcribed (by 11. 4.) ſo that the circumferences AB, BC, CD, DE, EA may be equal: And through the *points* A, B, C, D, E let GH, HK, KL, LM, MG be drawn touching the circle: and let F the center of the circle ABCDE be taken; and let FB, FK, FC, FL, FD be joined.

And

Book IV.



And becauſe the ſtraight line KL touches the circle ABCDE in the *point* C; and from the center F to the contact at C the *line* FC hath been drawn; therefore (by 18. 3.) FC is perpendicular to KL; therefore each of the angles at the point C is a right angle. For the ſame reaſon alſo the angles at the points B, D are right angles: and becauſe the angle FCK is a right angle; therefore (by 47. 1.) the *ſquare* of FK is equal to the *ſquares* of FC, CK: Certainly for the ſame reaſon alſo the *ſquare* of FK is equal to the *ſquares* of FB, BK; therefore (by com. not. 1.) the *ſquares* of FC, CK are equal to the *ſquares* of FB, BK; of which the *ſquare* of FC is equal to the *ſquare* of FB; therefore the remaining *ſquare* of CK is equal to the remaining *ſquare* of BK; therefore BK is equal to CK: And becauſe FB is equal to FC and FK common; certainly the two BF, FK are equal to the two CF, FK; and the baſe BK is equal to the baſe CK; therefore (by 8. 1.) the angle BFK is equal to the angle CFK; and the *angle* BKF; equal to FKC; therefore the *angle* BFC is double of KFC and BKC *is double* of FKC. Certainly for the ſame reaſon alſo CFD is double of CFL; and the *angle* CLD *is double* of CLF. And becauſe the circumference BC is equal to the *circumference* CD (by the conſt.); the angle BFC is alſo equal (by 27. 3.) to the *angle* CFD; and BFC is the double of KFC; and the *angle* DFC is the double of LFC; therefore the *angle* KFC is equal to CFL; certainly FKC, FLC are two triangles having the two angles equal to the two angles, each to each; and one ſide equal to one ſide; FC common to them *both*; therefore (by 26. 1.) they will alſo have the remaining ſides equal to the remaining ſides; and the remaining angle *equal* to the remaining angle; therefore the ſtraight line KC is equal to CL; and the angle FKC to FLC: and ſince KC is equal to CL; therefore KL is the double of KC. Certainly in the ſame manner HK will be demonſtrated to be double of BK: and becauſe BK has been demonſtrated to be equal to KC; and KL is the double of KC; and HK the double of BK; therefore HK is equal to KL; certainly in the ſame manner each of the *lines*

Book IV. GH, GM, ML will be demonſtrated to be equal to either of the *lines* HK, KL; therefore the pentagon GHKLM is equilateral. I ſay that it is alſo equiangular; for becauſe the angle FKC is equal to FLC; and HKL has been demonſtrated to be double of FKC; and KLM to be double of FLC; therefore HKL is alſo equal to KLM: certainly in the ſame manner each of the *angles* KHG, HGM, GML will be demonſtrated to be equal to either of the *angles* HKL, KLM; therefore the five angles GHK, HKL, KLM, LMG, MGH are equal to one another: therefore the pentagon GHKLM is equiangular; but it has been alſo demonſtrated to be equilateral; and it has been circumſcribed about the given circle ABCDE. Which was to be done.

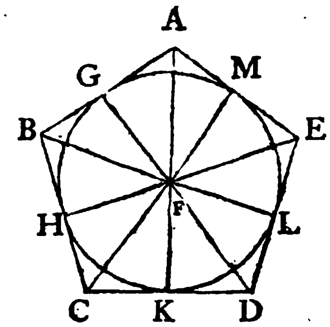
PROP. XIII.

To inſcribe a circle, in a given pentagon which is equilateral and equiangular.

Let ABCDE be the given pentagon, which is equilateral and equiangular: it is required to inſcribe a circle in the pentagon ABCDE.

Let each of the angles BCD, CDE be cut in halves, by each of the ſtraight lines CF, DF; and from the point F, in which the ſtraight lines CF, DF meet one another, let the ſtraight lines FB, FA, FE be drawn. And becauſe BC, CD are equal (by ſupp.); and CF common; certainly the two BC, CF are equal to the two DC, CF; and the angle BCF is equal to the angle DCF (by conſt.); therefore (by 4. 1.) the baſe BF is equal to the baſe DF; and the triangle BFC is equal to the triangle DCF; and the remaining angles are equal to the remaining angles, under which the equal ſides are extended; therefore the angle CBF is equal to the angle CDF: and becauſe the *angle* CDE is double of CDF; but CDE is equal (by ſupp.) to ABC; and the *angle* CDF to CBF; therefore CBA is the double of CBF; therefore the angle ABF is equal to FBC: therefore the angle ABC is cut in halves by the ſtraight line BF. Certainly in the ſame manner it will be demonſtrated, that each of the angles BAE, AED hath been cut in halves by each of the ſtraight lines FA, FE.

Let



Let the perpendiculars FG, FH, FK, FL, FM be drawn from the point F to the ſtraight lines AB, BC, CD, DE, EA; and becauſe the angle HCF is equal to KCF; and the right angle FHC is equal to the right angle FKC; certainly there are two triangles FHC, FKC having two angles equal to two angles; and one ſide equal to one ſide, viz. FC common to them *both*, extended under one of the equal angles; therefore (by 26. 1.) they will have the remaining ſides equal to the remaining ſides; therefore the perpendicular FH *is* equal to the perpendicular FK: certainly in the ſame manner it will be demonſtrated, that each of the *lines* FL, FM, FG is equal to either of the *lines* FH, FK; therefore the five ſtraight lines FG, FH, FK, FL, FM are equal to one another: wherefore the circle deſcribed with the center F and at the diſtance of *any* one of the *lines* FG, FH, FK, FL, FM will alſo paſs through the remaining points and will touch the ſtraight lines AB, BC, CD, DE, EA; on account of the angles at the points G, H, K, L, M being right angles; for if it will not touch them, but will cut them; it will happen that a *ſtraight line* drawn at right angles to a diameter from its extremity falls within the circle, which (by 16. 3.) has been demonſtrated to be abſurd: therefore the circle deſcribed with the center F and at the diſtance of *any* one of the *lines* FG, FH, FK, FL, FM will not cut the ſtraight lines AB, BC, CD, DE, EA; therefore it will touch them: let it be deſcribed as GHKLM.

Wherefore a circle has been inſcribed in a given pentagon, which is equilateral and equiangular. Which was to be done.

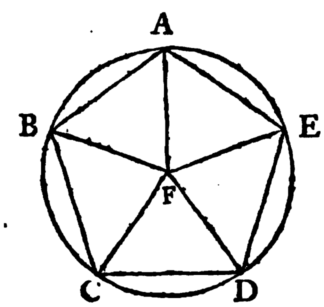
PROP. XIV.

To circumſcribe a circle about a given pentagon, which is equilateral and equiangular.

Let ABCDE be the given pentagon, which is equilateral and equiangular; it is required to circumſcribe a circle about the pentagon ABCDE.

Let

Book IV. Let each of the angles BCD, CDE be cut in halves by each of the ſtraight lines CF, DF; and from the point F in which the ſtraight lines meet, let the ſtraight lines FB, FA, FE be drawn to the points B, A, E; certainly in the ſame manner as in the *propoſition* before this it will be demonſtrated, that each of the angles CBA, BAE, AED hath been cut in halves by each of the ſtraight lines BF, FA, FE.



And becauſe the angle BCD is equal to the angle CDE; and the *angle* FCD *is* the half of BCD; and the *angle* CDF *is* the half of CDE; therefore alſo the *angle* FCD is equal to FDC: ſo that (by 6. 1.) the ſide FC is equal to the ſide FD. Certainly in the ſame manner it will be demonſtrated that each of the *lines* FB, FA FE is equal to either of the *lines* FC, FD: therefore the five ſtraight lines FA, FB, FC, FD, FE are equal to one another; wherefore the circle deſcribed with the center F and at ahe diſtance of *any* one of the lines FA, FB, FC, FD, FE will alſo paſs through the remaining points; and will be circumſcribed about the pentagon ABCDE, which is equilateral and equiangular; let it be circumſcribed and let it be ABCDE.

Wherefore a circle has been circumſcribed about a given pentagon, which is equilateral and equiangular. Which was to be done.

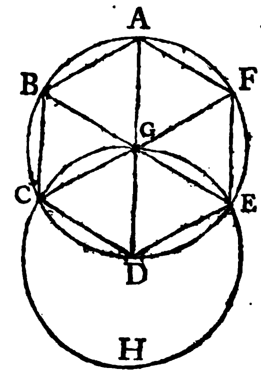
PROP. XV.

To inſcribe an equilateral and equiangular hexagon in a given circle.

Let ABCDEF be the given circle; it is required to inſcribe an equilateral and equiangular hexagon in the circle ABCDEF.

Let AD the diameter of the circle ABCDEF be drawn: and let G the center of the circle be taken; and with the center D and at the diſtance DG let the circle EGCH be deſcribed; and EG, CG being joined, let them be produced to the points B, F; and let AB, BC, CD, DE, EF, FA be drawn; I ſay that the hexagon ABCDEF is equilateral and equiangular.

For



For becauſe the point G is the center of the circle ABCDEF, the *ſtraight line* GE is equal to GD: again becauſe the point D is the center of the circle EGCH, the *line* DE is equal to GD; but GE has been demonſtrated to be equal to GD; therefore GE is equal to ED; therefore the triangle EGD is equilateral; and therefore the three angles of it EGD, GDE, DEG are equal to one another, ſince the angles at the baſe of iſoſceles triangles are equal to one another; and (by 32. 1.) the three angles of a triangle are equal to two right angles; therefore the angle EGD is the third *part* of two right angles: Certainly in the ſame manner it will be demonſtrated that the angle DGC is the third *part* of two right angles; and becauſe the ſtraight line CG ſtanding upon EB makes the adjacent angles EGC, CGB equal to two right angles; therefore the remaining *angle* CGB is the third *part* of two right angles; wherefore the angles EGD, DGC, CGB are equal to one another; ſo that (by 15. 1.) the vertical angles to them; *viz.* BGA, AGF, FGE are equal to EGD, DGC, CGB; therefore the ſix angles EGD, DGC, CGB, BGA, AGF, FGE are equal to one another; but (by 26. 3.) equal angles ſtand upon equal circumferences; therefore the ſix circumferences AB, BC, CD, DE, EF, FA are equal to one another; but (by 29. 3.) equal ſtraight lines are extended under equal circumferences; therefore the ſix ſtraight lines are equal to one another; therefore the hexagon ABCDEF is equilateral: I ſay it is alſo equiangular; for becauſe the circumference AF is equal to the circumference ED; let the common circumference ABCD be added; therefore the whole circumference FABCD is equal to the whole circumference EDCBA; and the angle FED ſtands upon the circumference FABCD; and the angle AFE upon EDCBA; therefore (by 27. 3) the angle AFE is equal to the *angle* DEF; in like manner it will be demonſtrated that the remaining angles of the hexagon ABCDEF, one by one, are equal to either of the angles AFE, FED; therefore the hexagon ABCDEF is equiangular; and it has been demonſtrated to be equilateral; and it has been inſcribed in the circle ABCDEF.

Wherefore an equilateral and equiangular hexagon has been inſcribed in a given circle. Which was to be done.

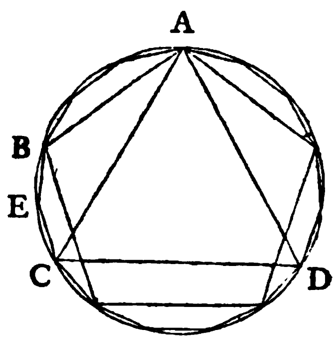
Book IV. Cor. From this *it is* manifeſt, that the ſide of a hexagon is equal to the *line* from the center of the circle.

And if we draw through the points A, B, C, D, E, F *ſtraight lines* touching the circle, an equilateral and equiangular hexagon will be circumſcribed about the circle, agreeable to what has been ſaid concerning the pentagon: And farther we ſhall inſcribe in a given hexagon and alſo circumſcribe a circle by the like *ſteps* as have been mentioned concerning the pentagon.

PROP. XVI.

To inſcribe an equilateral and equiangular quindecagon in a given circle.

Let ABCD be the given circle; it is required to inſcribe an equilateral and equiangular quindecagon in the circle ABCD.



Let AC, be inſcribed in the circle ABCD, the ſide of an equilateral triangle inſcribed in it; and let AB, be *inſcribed*, the ſide of an equilateral pentagon. Wherefore of what equal ſegments the circle ABCD is fifteen; of ſuch the circumference ABC, being the third *part* of the circle, will be five; and the circumference AB, being the fifth *part* of the circle, will be three; therefore the remainder BC will be two of theſe equal *ſegments*; let BC be cut in halves in the point E (by 30. 3.); therefore either of the circumferences BE, EC is the fifteenth *part* of the circle ABCD; therefore joining the ſtraight lines BE, EC; if we apply in the circle ABCD continually ſtraight lines equal to them (by 1. 4.) there will be inſcribed in it an equilateral and equiangular quindecagon. Which was to be done.

And in like manner as was done concerning the pentagon; if through the diviſions of the circle we draw *lines* touching the circle; an equilateral and equiangular quindecagon will be circumſcribed about the circle; and farther we ſhall inſcribe in a given quindecagon, which is equilateral and equiangular, and alſo circumſcribe a circle by the like *ſteps* as have been mentioned concerning the pentagon.

DISSERTATION VI.

IN the former diſſertation many things are not mentioned which a careleſs reader requires to be put in mind of, becauſe I conſidered that ſuch readers would ſtand in need of freſh admonitions, to induce them to read my obſervations; which determined me to ſteer a kind of middle courſe, by reducing my remarks to ſome general heads, and illuſtrating them by examples, which the reader is ſuppoſed to apply in all ſimilar circumſtances. Upon this principle it ſeemed needleſs to remark that the demonſtrations of the twelfth, thirteenth and fourteenth propoſitions of the fourth book, are more general than they appear to be at firſt ſight; for if the reader obſerve my general rule for the examination of every ſuppoſition; he will find no conſequence drawn from the number of ſides being five; it being only neceſſary that the inſcribed figures be equilateral and equiangular; which ſhews the demonſtrations to be much more general than is profeſſed in the propoſitions. In like manner, as it has been particularly remarked before, it ſeemed unneceſſary to deſire the ſtudent to obſerve the ingenious contrivance for cutting the angles of the figure in halves, in the thirteenth propoſition, having earneſtly recommended an attention to all ſuch indirect conſtructions: for in this inſtance if all the angles were cut in halves directly, it would be found no eaſy matter to prove that the lines will all meet in F; but by cutting any two of the angles in halves, which follow each other in order, the point F is fixed; and then joining this point, and the other angular points, it is eaſy to prove that theſe lines will cut all the other angles in halves. And whoever reads the former diſſertations with this allowance, will readily grant that I have been ſufficiently particular. I now proceed to a ſubject which will exer-

cise the reader's patience if he chuses to go along with me; for this dissertation will consist of a minute enquiry into the origin of our ideas of proportional magnitudes.

CHAP. I.

Of parts and multiples.

IN the first four books Euclid considers no other relation of magnitudes but their equality; at least when he speaks of any other, it is in a loose and undetermined manner, without ever considering how much greater or how much less the one is than the other. And this will be obvious to any one who chuses to turn his thoughts to the nature of his propositions; the common notion of whole and part therefore is sufficiently accurate for his purpose in these books. But the object which he has in view in this book, being to settle other relations of magnitudes besides their equality, makes it necessary to introduce, and consequently to define, a new idea of whole and part under the name of a multiple and part. Take two straight lines, one of them fixt and the other undetermined; by the third proposition of the first book two unequal lines being given, you may cut off a part from the greater equal to the less; and thus you may make a straight line, twice, three times, four times, or fivetimes &c. as long as another; the line originally fixt is called a part; and the line which you determine by this construction is called a multiple; they are relative terms, or as Euclid expresses it, a magnitude of a magnitude. The first thing therefore which the student has to settle in his own mind, is, what the magnitudes are and in what circumstances they must be, before this relation of multiple and part can take place; and not only so, but what the magnitudes are between which he can exhibit these relations scientifically from what has been demonstrated in the first four books. And first it is obvious that a straight line cannot be a part or multiple of a triangle; nor of the circumference of a circle; nor indeed can we consider one triangle as a part of another unless they be between the same parallel lines; nor the circumferences

rences of two circles; as part and multiple unless the circles be equal: but to settle this more particularly let us consider what magnitudes, we can multiply, that is measure exactly, and in what circumstances: Now we can multiply or measure exactly, two straight lines by the third proposition of the first book; that is we can take any straight line five, six &c. times; or if the supposition be that one straight line is five, six &c. times any other line; we can divide this multiple into its parts. But the reader is not to take my word for this; and the more instances in which he tries it, the more likely is he to understand what follows.

By multiplying (that is doubling, trebling &c.) the base of a triangle, you multiply the triangle, provided the part and multiple are between the same parallels; and the same may be said of parallelograms: so that here we begin to make our notions of magnitudes somewhat more complicated. It is said in the first book that triangles upon equal bases and between the same parallels are equal, but now we advance a step farther, and say that triangles upon double bases are double, and upon treble bases are treble &c; which transition the student who is desirous to have his ideas keep pace with his apparent progress in the science will do well to observe.

Angles, circumferences and sectors of equal circles may be multiplied by the principles contained in the twenty-sixth-seventh-eighth, and-ninth propositions of the third book: because by placing or applying equal straight lines in a circle you cut off equal circumferences; you make equal angles at the center or circumference; and it is also easy to prove that the sectors are equal: all this however the student must demonstrate; and I believe it will be found that these are all the magnitudes which can be multiplied according to Euclid's idea in his two first definitions; for it does not appear to me that it would be sufficient to have one triangle three times as large as another unless they be between the same parallel lines, to say that the one is a multiple of the other; because the definitions suppose that the multiple may be divided into its parts, which is what I understand by the word *measures*. But, all these, the student should examine by a particular construction, if he means to understand any part of this business; and I

am persuaded he will find, that ſtraight lines; triangles and parallelograms between the ſame parallel lines; angles, circumferences and ſectors of equal circles are all the magnitudes which can have the relation of multiple and part to one, who has no other principles to go upon but thoſe contained in the firſt four books.

It may be thought that particular inſtances explaining each of theſe circumſtances would have been uſeful; but the conſtruction of ſuch figures is extremely obvious, and it will be much more improving for the learner to deſcribe the figures for himſelf, becauſe he may look at figures already deſcribed, without comprehending what they are intended to illuſtrate, which can never be the caſe if the figures are properly deſcribed by himſelf. Now this being perfectly underſtood, and the proper meaning of the two firſt definitions determined; the ſtudent is next to endeavour to acquire a ready uſe of them; by reaſoning upon ſuch conſequences as follow moſt directly from them: let him ſuppoſe one magnitude to be a part of another and try what conſequences will follow; and the only direct conſequence is that the part will meaſure the multiple; but what is the meaning of this? Let the reader ſuppoſe two ſtraight lines which we ſhall call AB and C; and let C be a part of AB; then if we begin at the point A and (by 3. 1.) take of a part equal to C and from the remainder a part equal to C; we muſt at laſt fall in with the point B; becauſe, if this did not happen, C could not be a part of AB, contrary to the ſuppoſition; becauſe this is what I underſtand by meaſuring it: and the ſame conſequence follows if we ſuppoſe AB a multiple of C.

But it will be neceſſary to reaſon in the ſame manner upon each of the magnitudes, which have been mentioned as coming under theſe definitions of a part and multiple.

CHAP. II.

Of equimultiples.

THE word *equimultiple*, one might imagine has its meaning fully aſcertained, by the common acception of the two words of which it is made up; and yet I cannot tell how it happens, that it

it hardly ever conveys a distinct idea to the mind of a learner, as it is applied in this science; surely nothing seems more obvious than that it is to be understood of magnitudes taken an equal number of times; but I have generally found that there is some confused notion of equality of magnitude that accompanies the conception of equimultiples; than which nothing can possibly be more absurd; for it means only that the magnitudes are taken the same number of times; thus if any straight line, which you may call A be taken three times; and call this BC: and suppose any angle to be trebled; we say that BC and this angle, are equimultiples of the line A and of the angle which was taken three times; or to be more particular (turn to the figure for the first proposition of the sixth book) the straight line HC and the triangle AHC, though they can have no connexion with each other as to magnitude, are nevertheless equimultiples of the straight line BC and of the triangle ABC: and again (see the figure to the thirty third proposition of the sixth book) the circumference EN and the angle EHN, though magnitudes of quite different kinds, are equimultiples of the circumference EF and of the angle EHF; or in short this expression means no more, but that when the multiples are divided into their parts, the number of divisions in each is the same; as in the instances above, HC is divided into three parts each equal to BC; and the triangle AHC into three parts also each equal to the triangle ABC; these equimultiples therefore agree in nothing but that they can be divided into the same number of parts; and this is the most direct consequence which will follow from the supposition of their being equimultiples. But in order to understand the meaning of this term, and the two first definitions it will be proper for the reader to examine the three first propositions together with the fifth and sixth of this book.

CHAP. III.

Containing further illustrations of the first and second definitions.

THE reader will perceive, if he has gone through the different steps of the constructions which I have recommended, that all those

those magnitudes which we can multiply; are doubled, trebled &c. by the assistance of a straight line; so that whatever can be shewn to be a property of, a multiple, equimultiples or parts of a straight line the same may be readily transferred to triangles or parallelograms, between the same parallel lines; to angles; circumferences and sectors of equal circles; which I contend are all the magnitudes which we can multiply, agreeable to Euclid's two first definitions; and probably for this reason the constructions and demonstrations in this fifth book are confined to straight lines only; although the enunciations are expressed of magnitudes in general. Let us suppose that the reader has examined the three first propositions and the fifth and fixth; only observing, that Euclid is here demonstrating properties of straight lines, I shall now proceed to make some remarks upon them in order. And first it is said that AB is a multiple of E; *equimultiple you mean*; no, not so fast let us examine one thing at a time; I say that the supposition is; AB is a multiple of E; well let us see what consequence follows from this; cut off from AB the line AG equal to E; and by proceeding thus you must fall in with the point B; that is I thus divide AB into parts equal to E; or I measure AB; take care to distinguish between this, *and the supposing it divided*; for if the reader be given to suppositions of this kind, he had better suppose the whole book demonstrated. In like manner CD by the supposition is a multiple of F; and the same consequences follow; namely that in the same manner I divide CD into parts equal to F: But farther by the supposition AB and CD are equimultiples of E and F; let us see what consequence follows from this part of the supposition; nothing more than this, that after I have divided the multiples into magnitudes equal to the parts, the number of divisions in both is the same; for certainly if AB be the double of E and CD the double of F; AB contains E twice and CD contains F twice; and the same may be said of any other equimultiples; and the use made of it in this proposition is, that if you take away the parts one by one, when you have done with one multiple all the other equimultiples must be exhausted at the same time.

But farther, this proposition should be examined still more particularly; because, although this may seem needless for those to whom

whom Euclid writes, I know it to be essentially necessary for those to whom this dissertation is addressed. The student is therefore to bring this conclusion more directly to the definitions, at least to take such steps as seem most likely to carry his attention up to the definitions. For this purpose let him draw two indefinite straight lines; and cutting off from one of them a line equal to E and F together; he is then to proceed in the same manner and cut off from the other a line equal to AG and CH together; and from the remainder a line equal to GB and HD together; and so on until he has made a line equal to AB and CD together: He is next to demonstrate that the line, made equal to E and F together, will measure the line made equal to AB and CD together; and when this is done he can say, that the one line is a part or multiple of the other: But after all it remains to be proved that the one line is contained in the other, just as often as E is contained in AB; and then he may say that AB; and AB, CD together are equimultiples of E; and of E, F together. Let him, after this point is settled to his full conviction, proceed in the same manner with three multiples &c. until he can say upon good grounds if there be any number of magnitudes &c.

The second and third propositions must be perfectly intelligible to any one who has gone the progress which I have recommended upon the first. But before entering upon the fifth it will be proper for the reader to observe that we are not as yet taught how to take even a third part of a straight line, much less how to take any part: indeed we can cut a straight line in halves; by which construction we may take a half, a fourth part, and an eighth part &c. but nothing more: the construction therefore in the fifth proposition is impracticable; for if AB be three times CD; I cannot take a third part of BE but upon the supposition that EB is a multiple of FD; which is taking for granted the very thing to be proved: if a line be a multiple of another I can divide it into its parts; but not otherwise; which makes a different construction necessary here, from that given in the text; but sufficiently obvious from the construction in the next proposition: for producing BA directly forward, we may cut off lines equal to FD, until we have the same multiple of FD, that AB is of CD; and call this AG; then

then by the first proposition EG is the same multiple of CD that AB is of CD, therefore (by com. not 6.) AB and EG are equal; take away AE which is common, and EB is equal to AG; therefore EB is the same multiple of FD that AB is of CD.

It is not easy to conjecture how this blunder could have crept in, unless by the deception of a particular figure: for if AB be the double of CD, the construction is practicable, because then I divide EB in halves; and take CG equal to the half of it; and then indeed AB is the same multiple of GF as of CD, therefore GF and CD are equal &c. And it is obvious that if AB be taken four times CD the same construction will hold; but it fails, if AB be taken three times, for then I have no principle upon which I could pretend to divide EB; unless I should measure it by FD, which is taking the very thing for granted which we want to demonstrate, namely that EB is a multiple of FD; and not only a multiple of it, *but the same multiple of it*, that AB is of CD; that is, I set out with a supposition that CG is equal to FD in order to prove them to be equal; which is certainly very bad reasoning if it deserve the name. I have been the more particular in explaining this, because it seems well calculated to illustrate this doctrine of parts and multiples.

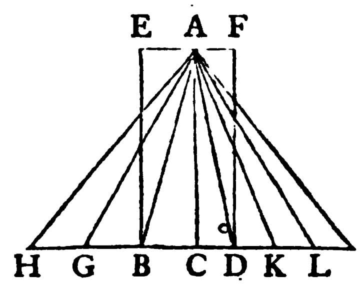
CHAP. IV.

Containing a comparison of the multiples of four magnitudes.

LET ABC, ACD be two triangles between the same parallel lines; let equimultiples of the base BC and of the triangle ABC be taken; and also any equimultiples whatever of the base CD and of the triangle ACD; I say that if the multiple of BC exceed the multiple of CD; the multiple of the triangle ABC will exceed the multiple of the triangle ACD; if equal, equal; and if less, less.

Here the reader is to fix his attention upon four magnitudes, which he may distinguish thus, by calling BC the first, CD the second, the triangle ABC the third, and the triangle ACD the fourth; according to our supposition then, we are to take equimultiples

multiples of the first and third at a venture; and also any equimultiples whatever of the second and fourth. But observe we do not compare the *equimultiples* with one another; only the multiple of the first, with the multiple of the second, which are both straight lines; and the multiples of the third and fourth, which are both triangles. And first let us exhibit any multiple of BC the first at a venture; produce CB indefinitely and cut off as many lines as you please each equal to CB; as BG, GH &c; and let CH be a multiple of BC taken at a venture, which it is if we draw no consequence from CH's containing CB any particular number of times. Now we are to take the triangle ABC the same number of times; and this is performed by joining AH, for the triangle AHC contains the triangle ABC just as often as CH contains CB; which may be proved by joining AG; for as many lines CB, BG, GH as there are, just so many triangles are there; and because these straight lines are equal; (by 38. 1.) the triangles are equal of which these lines are the bases; and the number of triangles, and the number of straight lines being the same; CH contains CB just as often as the triangle AHC contains the triangle ABC; therefore CH and the triangle AHC are equimultiples of CB and of the triangle ABC. Again we are to take equimultiples any whatever of the second and fourth, of CD and of the triangle ACD: produce CD directly forward, and take DK, KL each equal to CD; and in the same manner CL is any multiple of CD; and the triangle ACL the same multiple of the triangle ACD. The construction being thus finished let us proceed to the demonstration.



Now I say that whenever CH is less than CL; the triangle AHC is less than the triangle ACL; because I may cut off from CL a part equal to CH; and then joining the extremity of that line to the point A; there will be a triangle, equal to ACH within the triangle ACL; therefore when CH is less than CL, the triangle AHC will always be less than the triangle ACL; and if CH be equal to CL, then (by 38. 1.) the triangle AHC is equal

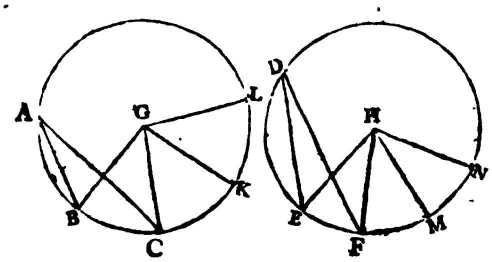
to ACL: and laftly if CH be greater than CL, it may in like manner be proved that the triangle AHC is greater than the triangle ACL: and all this will be true according to any multiplication whatfoever. And if for the fake of diftinctnefs we call BC the firft; CD the fecond; the triangle ABC the third; and the triangle ACD the fourth; we have this general conclufion; if two fuch triangles be between the fame parallel lines; and if you take any equimultiples whatever of the firft and third; and alfo any equimultiples whatever of the fecond and fourth; whenever the multiple of the firft exceeds the multiple of the fecond the multiple of the third will always exceed the multiple of the fourth: and when equal, equal: and when lefs, lefs. Which was to be demonftrated.

The reader is now to proceed by himfelf and prove the fame thing of the equimultiples of BC and of the parallelogram CE; and the equimultiples of CD and of the parallelogram CF; that is, that whenever the multiple of BC exceeds the multiple of CD; then the multiple of the parallelogram EC will exceed the multiple of the parallelogram CF: and when equal, equal: and when lefs, lefs.

Here our reafoning feems to become fomewhat more complicated than in the firft four books: but if we attend carefully to the meaning of the terms, we fhall find very little difference: for the multiple is compofed by a repetition of the *fame* magnitude; which may therefore be confidered only as a comparifon of two diftinct things: the term *equimultiples* implies only a divifion of the magnitudes into the fame number of equal parts: and even in this laft comparifon, we have only two magnitudes to compare with one another, as to their being greater; equal; or lefs. So that if the ftudent but chufes to prepare himfelf properly, he ftands upon the fame firm ground here, as in the firft four books: for the magnitudes are the fame, only the mode of comparifon is new: Which muft be made familiar by particular inftances. Therefore let us fuppofe two equal circles ABC, DEF; and BGC, EHF angles at their centers, ftanding upon the circumferences BC, EF: I fay, if you take equimultiples of the circumference BC and of the angle BGC at a venture; and alfo any equimultiples of the circumference

cumference EF and of the angle EHF: if the multiple of BC exceed the multiple of EF; I ſay the multiple of the angle BGC will always exceed the multiple of the angle EHF; and if equal, equal: and if leſs, leſs.

For take any multiple of the circumference BC, by placing or applying in the circle ſtraight lines, equal to a ſtraight line joining B and C; and thus making the circumferences CK, KL equal to the circumference BC (by 28. 3.): and thus BL may repreſent any multiple of the circumference BC: but becauſe the circumferences BC, CK, KL are equal to one another, (by 27. 3.) the angles BGC, CGK, KGL are equal to one another; therefore the angle BGL is a multiple of the angle BGC; but moreover it is the ſame multiple of this angle, that the circumference BL is of BC; therefore the circumference BL and



the angle BGL are equimultiples of the circumference BC and of the angle BGC: And in the ſame manner we may ſhew that the circumference EN and the angle EHN are equimultiples of the circumference EF and of the angle EHF: now ſuppoſing as in the laſt inſtance, the circumference BC to be the firſt; the circumference EF the ſecond; the angle BGC the third; the angle EHF the fourth; I ſay that whenever the circumference BL, the multiple of the firſt, exceeds the circumference EN, the multiple of the ſecond; then the angle BGL the multiple of the third, will exceed the angle EHN the multiple of the fourth; for, becauſe BL exceeds EN, I may take of from BL a part equal to EN; by placing in the circle a ſtraight line equal to the ſtraight line joining the points E and N; and can thus prove that a part of the angle BGL is equal to EHN: and alſo if BL be equal to EN the angle BGL is equal to the angle EHN; and in the ſame manner it may be proved that if the multiple BL be leſs than EN, then the angle BGL is leſs than the angle EHN; and that all this will happen according to any multiplication whatever &c.

But here a difficulty ſeems to occur, becauſe the multiple may be ſo taken that BG and GL will become one ſtraight line. Alſo in taking a multiple of a ſtraight line; there is obviouſly no bounds ſet to the operation; which there ſeems to be in taking a multiple of a circumference; though this may be rendered indefinite by deſcribing as many equal circles as we pleaſe. Now what I take to be Euclid's meaning, as being moſt conſiſtent with the common notion of an angle, is this, that in comparing the multiples of the circumferences, all the ſemicircumferences are to be neglected; and in comparing the multiples of the angles at the center; we are only to conſider, that part of the angle which exceeds two right angles; which makes the conſtruction and figure here given general. I would recommend it to the ſtudent, to demonſtrate the ſame thing, of the multiples of the circumferences and ſectors of equal circles compared together; as alſo, of the multiples of the angles at the center and ſectors.

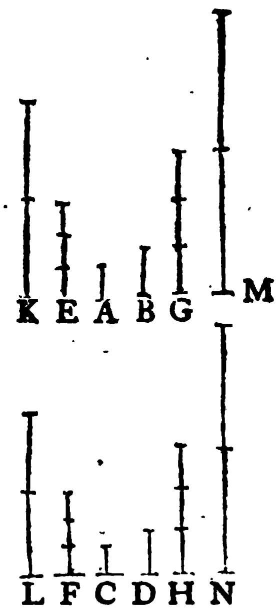
CHAP. V.

The ſame ſubject continued.

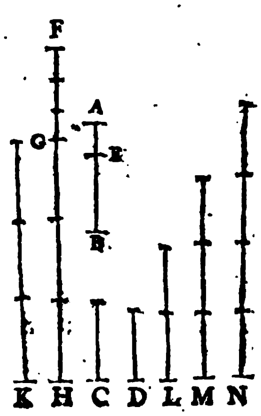
LET there be four magnitudes A, B, C, D; and let E, F be any equimultiples of A and C the firſt and third: and G and H any other equimultiples of B and D the ſecond and fourth: And let any equimultiples of A and C; and any equimultiples of B and D be ſo particularly connected with one another; that whenever the multiple of A exceeds the multiple of B; the multiple of C will always exceed the multiple of D; and when equal, equal; and when leſs, leſs: I ſay, that if any equimultiples of E and F be taken; and any whatever of G and H; it will always happen that when the multiple of E exceeds the multiple of G; the multiple of F will exceed the multiple of H; and when equal, equal; and when leſs, leſs. That is take K and L any equimultiples whatever of E and F; and alſo M and N any equimultiples whatever of G and H; I ſay that when K exceeds M; L will always exceed N; and when equal, equal; and when leſs, leſs.

THE

THE DEMONSRATION. Because E and F are equimultiples of A and C; and K and L equimultiples of E and F; therefore, by the third proposition of the fifth book, K and L are equimultiples of A and C: Again because G and H are equimultiples of B and D; and M and N of G and H; therefore (by 3. 5.) M and N are equimultiples of B and D: but *the supposition is*, that if there be any equimultiples of A and C; and any whatever of B and D; if the multiple of A exceed the multiple of B; than the multiple of C will always exceed the multiple of D; and when equal, equal: and when less, less; therefore because it has been demonstrated that K and L are equimultiples of A and C and M and N of B and D; it follows that when K exceeds M; L will exceed N; and when equal, equal: and when less; less. Which was to be demonstrated.



Let AB, C and D be three such straight lines; that any equimultiples whatever of AB and C compared with any multiple of D; may be thus related; that whenever the multiple of AB exceeds the multiple of D; the multiple of C shall also exceed the multiple of D; and when equal, equal: and when less, less: I say if this be the case AB is equal to C. For if not, one of them must be greater, which suppose to be AB. Cut off a part from AB equal to C; and let this be BE: multiply AE their difference until it exceed D: and let this multiple be FG; therefore FG is greater than D: make GH the same multiple of EB; that FG is of AE; therefore (by 1. 5.) FH is also the same multiple of AB: make K the same multiple of C; therefore FH and K are equimultiples of AB and C. Make L the double of D, and so on, until we come to the multiple which is the next greater than K: let this be N: M therefore is not greater than



K.:

K: but K is equal to GH; because EB is equal to C; therefore M is not greater than GH; but FG is greater than D (by const.) wherefore FH is greater than M and D together, that is N; therefore FH is greater than N; and K is not greater than N: and FH and K are equimultiples of AB and C and N is a multiple of D; but by the supposition any equimultiples of AB and C were so related to any multiple of D; that whenever the multiple of AB exceeds the multiple of D; the multiple of C also exceeds the multiple of D; but the contrary has been proved when we suppose AB unequal to C; therefore the multiples being according to the supposition AB cannot be unequal to C. Which was to be demonstrated.

Let there be six magnitudes, or rather straight lines A, B, C, D, E, F whose multiples are thus related: suppose any equimultiples of A and C to be taken, and also any equimultiples whatever of B and D; and farther whenever the multiple of A exceeds the multiple of B; let the multiple of C always exceed the multiple of D: and again let there be some equimultiples of C and E and also of D and F such that the multiple of C exceeds the multiple of D; but the multiple of E does not exceed the multiple of F: I say that there are some equimultiples of A and E; and some of B and F such that the multiple of A exceeds the multiple of B; when the multiple of E does not exceed the multiple of F.

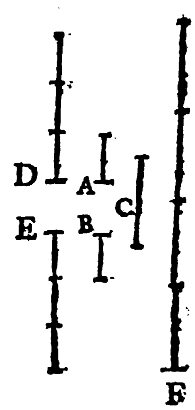
Suppose G and H to be such equimultiples of C and E; and K and L such equimultiples of D and F; that G exceeds K but H does not exceed L: now because by the supposition G and K are given, I can find what number of times C and D are contained in them by measuring them, that is by cutting off parts equal to C and D. Take M the same multiple of A that G is of C; that is, which H is of E: again take N the same multiple of B which K is of D; that is, which L is of F.

Demonstration. Now by the supposition A, B, C, D are so related to one another that any equimultiples of A and C; and also any equimultiples of B and D being taken; if the multiple of A exceed the multiple of B the multiple of C exceeds the multiple of D: but M and G are (by const.) equimultiples of A and C; as also N and K of B and D; therefore if M exceed N; G exceeds

exceeds K: but G does exceed K (by supp.); therefore M exceeds N: but H does not exceed L (by supp.); wherefore M exceeds N but H does not exceed L; and M and H are equimultiples of A and E; and N and L of B and F. Wherefore &c. Which was to be demonstrated.

If D and E (see the next figure) be such equimultiples of A and B; and F such a multiple of C, that D exceeds F but E does not exceed F; then I say that A is greater than B.

For because D exceeds F and E does not exceed F; therefore of consequence D exceeds E: but because D and E are equimultiples of A and B and D exceeds E; therefore any part of D will exceed the same part of E; that is if one of the magnitudes be greater than another, the third &c. part of the one will exceed the third &c. part of the other; therefore A is greater than B.



Again if F be such a multiple of C; and D and E such equimultiples of A and B, that F exceeds E but F does not exceed D; then, I say that B is less than A.

For, because F exceeds E but F does not exceed D; therefore E is less than D; therefore any part of E will be less than the same part of D; but A and B are the same parts of D and E therefore B is less than A. Which was to be demonstrated.

CHAP. VI.

Containing an explanation of the remaining definitions.

IF what has been said, in this dissertation, be properly attended to, there will be little difficulty in understanding the meaning of the third definition; which has been the occasion of so much trouble to the commentators; and which they have been so very unsuccessful

unſucceſsful in their attempts to explain. For by proceeding according to the ſpecimen here given, it is very obvious that all the properties of magnitudes, which are mentioned in the fifth book, may be demonſtrated from the two firſt definitions, without any mention of the term ratio. But to ſave the trouble of ſuch circumlocutions; our author gives particular names to the different ſuppoſitions; and this is the true origin of the following definitions, from the ſecond. Now ſays he, in the third, any certain mutual habitude of magnitudes of the ſame kind, conſidered according to this *quantuplicity* or multiplicity; or according to this relation of parts and multiples, I call *ratio*. And I believe I may ſay that this definition has never been rightly underſtood ſince the greek became a dead language. *Barrow* calls it a metaphyſical definition; but it appears from this, that it is a mathematical one. He ſays that ſuch definition was equally neceſſary in the ſeventh book, when ſpeaking of the ratio of numbers, but it appears from this explanation, that ſuch a definition could have no place there.

When any equimultiples of the firſt and third, and any equimultiples whatever of the ſecond and fourth, have this relation to one another, that when the multiple of the firſt exceeds the multiple of the ſecond; the multiple of the third always exceeds the multiple of the fourth; or when equal, equal; or when leſs, leſs; then he ſays the firſt has the ſame ratio to the ſecond which the third has to the fourth: But if the multiple of the firſt exceed the multiple of the ſecond, when the multiple of the third does not exceed the multiple of the fourth; he then ſays that the firſt has a greater ratio to the ſecond than the third has to the fourth. The ſixth, eighth, and ninth definitions are ſufficiently obvious. When three magnitudes are proportionals; the firſt is ſaid to have to the third the duplicate ratio, of that which it has to the ſecond. But this definition and the next I ſhall have occaſion to explain afterwards.

When we ſay that the firſt has the ſame ratio to the ſecond which the third has to the fourth; the firſt and third are the antecedents; and the ſecond and fourth are called the conſequents. Now the antecedents are called homologous terms, or terms of like ratio; and ſo are the conſequents alſo. This diſtinction is neceſſary for ſeveral

ſeveral reaſons; in the following definitions, and in fixing the equal angles of ſimilar triangles &c, and therefore ought to be particularly attended to.

The following definitions will be ſufficiently underſtood, when the particular propoſitions are read, wherein it is proved that the magnitudes will have the ſame ratio, the firſt to the ſecond and the third to the fourth, after they have undergone the changes, mentioned in theſe definitions. I know there is a way of explaining theſe definitions by numbers, which is nothing to the purpoſe; for Euclid is not ſpeaking of numbers at preſent but of magnitudes. However as this ſupplies the place of all the knowledge which this book contains to many, who are not ſo ſcrupulous as to require a demonſtration; nor ſo attentive as to conſider whether they are talking about ſomething or nothing, to gratify ſuch indolent readers, I have preſented them with the following ſcheme, ſerved up in the true French taſte.

$$15:5::30:10\begin{cases}\text{Alt.} & 15:30::5:10\\ \text{Inv.} & 5:15::10:30\\ \text{Comp.} & \begin{cases}15+5:5::30+10:10\\ 20:5::40:10\end{cases}\\ \text{Div.} & \begin{cases}15-5:5::30-10:10\\ 10:5::20:10\end{cases}\\ \text{Conv.} & 15:15-5::30:30-10\\ & 15:10::30:20\end{cases}$$

$$\begin{array}{l}\text{By equal. ord.}\begin{cases}3\,.\,6\,.\,18\\ 5\,.\,10\,.\,30\end{cases}\\ \hline 3:18::5:30\end{array}\qquad\begin{array}{l}\text{By equal. pert.}\begin{cases}7\,.\,21\,.\,42\\ 8\,.\,16\,.\,48\end{cases}\\ \hline 7:42::8:48\end{array}$$

I hope, from what has been ſaid, the reader underſtands the meaning of theſe definitions. The triangle, rectangle and circle have been conſidered as inſtruments of inveſtigation; but this doctrine of *ratios* is to be regarded as a new mode of compariſon, very extenſive, in its conſequences, giving us a wonderful command over the magnitudes, which we have already conſidered, by diſcovering a great many properties far beyond the reach of the former method of compariſon. And although a great deal of pains has been taken to render our ideas upon this ſubject, confuſed, indiſtinct and nothing; yet it is wonderful, when properly examined, what a ſimplicity there is in the principles upon which this method of compariſon is founded; the conſtructions are all performed (by 3. 1.) and requires no ſuch *apparatus* as is neceſſary even

to prove one line to be equal to, or ſhorter than another, when we cannot make uſe of the circle.

CHAP. VII.

Of the arrangement of the propoſitions.

THE reader who has been accuſtomed to have his head filled with numerical ideas as explanatory of the propoſitions contained in this book, will perhaps be ſurprized to hear me affirm that he has taken a great deal of pains to fill his head with abſurdities. For I am perſuaded that if a ſcheme were to be thought of, for depriving a man of his reaſon, ſtill keeping up an impreſſion upon his mind that he was a rational creature, there could not be a more effectual method, than to ſet him upon reconciling the demonſtrations of this book to numerical ideas. Indeed the twiſting ropes of ſand would be a rational employment when compared with this.

All numerical reaſoning proceeds upon the ſuppoſition, that the unit is the ſame. But Euclid is not yet prepared for this confinement, which I ſhall prove particularly, upon the ſeventh book. He does not carry on his demonſtrations in the firſt four books, upon the ſuppoſition that the ſides of his triangles, or parallelograms or the radius of his circle, are three or four feet long; or as having a reference to any kind of meaſure. Nor is his reaſoning in this book leſs general. His *ratio* is a mutual habitude of magnitudes according to quantuplicity, or according to the doctrine of parts and multiples. The queſtion is not; is the part *one*; but is it a fixt magnitude? and farther, if you mean to reaſon upon it according to this doctrine of *ratios*; is it a magnitude that you can multiply? that is, that you can double; for the whole operation conſiſts only of doubling. It would not be more abſurd, to ſuppoſe that the nature of things, or human imagination does not allow of any triangle but an equilateral one, and nevertheleſs to try to prove that the ſquare of one ſide of a triangle might be equal to the ſquares of the other two; than to attempt the proof of ſome of the ſimpleſt properties of triangles, parallelograms and circles upon the ſuppoſition that all magnitudes can be expreſt by numbers.

It

It is to be hoped what has been said will be sufficient to induce the reader to pay a proper attention to the structure of the demonstrations made use of in this book, which have a wonderful elegancy and force even when taken singly; but more particularly when the arrangement of the whole is considered: for the better understanding of which I would advise the reader to divide the book into four portions. The nature of *ratios* shews the propriety and necessity of introducing the properties of equimultiples first of all; and although the fourth proposition is not absolutely confined to its place; yet as it depends upon the third, it would be difficult to give it a more advantageous position. The first six propositions therefore will make the first portion. Again, we may have a very distinct notion of this *ratio* of magnitudes without knowing what the magnitudes themselves are directly: Here therefore some rule or test is necessary for enabling us to get at the common relations of greater, equal and less, when we have only this *ratio* to guide us. The student is therefore to consider the seventh, eighth, ninth and tenth propositions, as making a distinct section, and as introduced for the very purpose of settling this point.

It might appear to a superficial observer, who had just come from proving that if four magnitudes are proportionals, they will be proportionals by inversion; that a similar method of reasoning might be applied to shew that they will be proportionals by alternation. But it happens here, that the equimultiples, of the first and third, of the magnitudes which we want to prove to be proportionals; are not equimultiples of the second and fourth of those magnitudes which are proportionals by the supposition; but of the first and second; which destroys all prospect of succeeding according to that plan. And indeed this demonstration is no easy matter, for all the propositions from the tenth to the seventeenth may be considered as introduced to demonstrate; that magnitudes will be proportionals by alternation. And this will make up the third portion according to the division above mentioned. And the remaining part of the book may be considered under one head, in which is proved that magnitudes will be proportionals; by composition division, conversion, and by the two methods of reasoning by *equal distances*. And thus I think a very distinct and comprehensive view

of the ſubject of this book may be taken, and ſuch an one as ſeems likely to fix it in the memory.

CHAP. VIII.

Containing ſome remarks on the demonſtrations.

AGREEABLE to the rule already laid down, inſtead of remarks upon particular demonſtrations, I ſhall reduce what I have to ſay under diſtinct heads, which the reader may apply as he goes along; but firſt it ſeems neceſſary to be a little particular, as to the conſtruction of the thirteenth propoſition. Indeed there is nothing which ſeems to be ſo prevailing an error, with regard to this book, as a neglect of the conſtructions. And yet this is a thing which Euclid is, every where, more particularly attentive to. He ſometimes leaves his reader to draw the conſequence from a conſtruction, when it is ſufficiently obvious, and cannot be done ſhortly and elegantly; an inſtance of which we have in the firſt propoſition of the third book; where there are two conſtructions, but he reaſons directly, only from one of them, and then joins both their conſequences in his concluſion: for the demonſtration only proves that the center muſt be in the perpendicular; for no abſurdity will follow from the reaſoning, by ſuppoſing the point G to be any where in the line CD; not even if we ſuppoſe it to be at the circumference; for it will not then be abſurd to ſuppoſe the angle GDB equal to CDB; we have therefore Euclid's authority for drawing conſequences in this manner; and I have not the leaſt doubt of this demonſtration being the genuine one given by Euclid; becauſe to have made it quite regular the two conſtructions muſt have been ſeparated; and then it would not have been one, but two diſtinct propoſitions; in the firſt of which it was to be proved that the center muſt be in the perpendicular; and in the ſecond, that this perpendicular cut in halves would find the center; which is a much greater formality than the nature of the problem requires, although the ſtudent ought to examine it in this manner. But in this thirteenth propoſition the caſe ſeems to be very different, and conſequences are drawn from a conſtruction which I believe

believe it will be found very difficult to perform. *Simson* in his note upon this proposition says; "In Commandine's, Brigg's and "Gregory's translations, at the beginning of this demonstration, "it is said, *and the multiple of C is greater than the multiple of D; "but the multiple of E is not greater than the multiple of F*; which "words are a literal translation from the greek: But the sense evi-"dently requires that it be read, so that the multiple of C be "greater than the multiple of D; but the multiple of E be not "greater than the multiple of F." Now I am so much of a contrary opinion, that it appears to me that this change destroys every vestige of the true construction of this proposition. My notion is that the proper equimultiples are given me by the hypothesis; and the multiples and magnitudes being given, I can measure the one by the other; and consequently know how many times the magnitudes are taken; from which the construction is obvious: but if I have nothing to direct me but the magnitudes themselves; concerning which also there is no supposition of greater or less, I should be glad to know in such circumstances, how this construction is to be performed; I know it is said *let them be taken*, which if I were to add my conjectures after the manner of some commentators, I would suspect to be the correction of some hasty editor who did not understand the demonstration.

In proving four magnitudes to have the same ratio, the first to the second which the third has to the fourth; it is necessary that the equimultiples of the first and third be *any whatever*; and also that the equimultiples of the second and fourth be taken at a venture. Now it is obvious that if they be taken as often as one determinate magnitude contains another, they cannot be *any whatever*; neither is the sum of two multiples, each of which has been taken at a venture, to be considered as any multiple whatever; because if in one case it should happen to be taken three times, and in the other five times; these multiples added together, would not *happen* to be eight times that magnitude, but must really be eight times the magnitude, not accidentally but consequentially: again when a multiple of a magnitude is taken, and then a multiple of that multiple as in the third proposition; this last is not *any multiple whatever* of the first magnitude; for instance the first happens to

to be taken three times, and this multiple again five times; now the last multiple does not *happen* to be fifteen times the first, magnitude, but must be really so. And this will be sufficient to direct the student to the proper use of the phrase *any whatever*.

Simson's remarks upon the eighteenth proposition are very ingenious; for the possibility of the fourth proportional is not to be taken for granted: because this is a very different notion from that which superficial people are apt to confound with it: for when two magnitudes are made the subject of contemplation, their *existence* is a part of the supposition; and if they do exist, it must be under one or other of these forms; A must be equal to B, or greater, or less; and if I prove that A cannot be greater than B nor less, I certainly demonstrate that A is equal to B. But it is very different where the *existence* of the magnitude may be called in question, which is very reasonably done in the present instance; for it may be said to Euclid you have confined yourself so much by your definition of *the same ratio*, that a fourth proportional may be really impossible according to your definition; but when you have once shewn the possibility of it, I will then admit your reasoning. It is very certain, that we have sufficient principles, before this proposition, for finding a fourth proportional to any three given straight lines, which would make the demonstration unexceptionable as we have it at present; by proceeding thus; to AB, BE and CD find a fourth proportional, which must be greater, equal or less than FD &c; but this would be mixing things together, which our author certainly intended to keep separate. But to conclude this chapter, whoever has a proper notion of a part and multiple; and knows what is required to prove that four magnitudes have the same ratio the first to the second which the third has to the fourth, can never be mistaken in forming a judgement of the demonstrations contained in this book. Euclid constantly supposes his magnitudes to be straight lines as is obvious from his constructions and demonstrations; so that there is no occasion to suppose the magnitudes in any proposition to be of the same kind; and indeed those critics who have made such alterations, if they had understood their business, ought at least to have changed the enunciations of half the propositions in the book.

CHAP.

CHAP. IX.

Of the subject and arrangement of the sixth book.

OUR author having laid down this general doctrine of parts and multiples proceeds to apply it to the investigation of such properties of the triangle, rectangle and circle as could not be obtained by the former method of comparison. For what is delivered in the fifth book itself cannot, strictly speaking, be considered as properties of triangles, parallelograms &c. but only as properties of parts and multiples of such magnitudes as come under the two first definitions. But in this book he treats of the triangle as such, and shews when triangles have the same altitude, they have the same ratio to one another, as their bases. And here the thoughtless reader will do well to observe; that every triangle may have its altitude exprest by three different straight lines, as each side can be the base: for the altitude of a triangle is a relative term, and has no meaning until the base is fixt. From this he demonstrates that if the two sides of a triangle be cut by a straight line parallel to the third side, they will be cut in the same proportion; this is the fundamental principle which runs through the whole of this book; and ought for this reason to be examined in every point of view. For after the reader has considered this property according to its most obvious acceptation, he will find it more general than he suspected, by making it a property of two indefinite straight lines intersecting one another, and cut in any manner by two parallel lines.

It has been mentioned already that the triangle is the great instrument of geometrical investigation, and Euclid's first object in this book is to lay down those principles upon which the similarity of triangles depends; and by this acquisition the power of the science becomes astonishingly great, as it is difficult to say what cannot be done by it that is really practicable; the judicious reader therefore will examine the first eight propositions with that attention which the importance of the subject requires. The five following propositions teach how to divide a straight line into any number of equal parts; to cut a given line in the same proportion, as

as another has been cut; to find a third, fourth and mean proportional; which being practical principles, should be made habitual. The next four propositions are all upon the same subject, and that a very curious one, namely to fix the equality of the parallelograms and triangles from the ratio of the sides and conversly. And here the indolent reader, should be required to shew what use is made of the equality of the angles, or where the demonstration would fail if that circumstance were omitted. The nineteenth and twentieth propositions are of great importance and should be carefully studied: it will assist a beginner very much to consider them as problems, omitting the term duplicate ratio entirely: Thus the ninteenth may be exprest in this manner; two similar triangles being given, to find two straight lines which shall have the same ratio to one another which the triangles have. And BC and BG will be the two straight lines required. The reader may know whether he understands the proposition, by taking a third proportional to EF and BC, and compleating the construction and demonstration. The twentieth may be worded thus; to find two straight lines which shall have the same ratio to one another which two similar polygons have; and then to prove that the same two straight lines, will have the same ratio to one another which the similar triangles, into which the polygons are divided, have to one another. And it will also be convenient, according to the same plan, to consider the twenty third as a problem; which may be exprest in this manner; to find two straight lines which shall have the same ratio to one another which two equiangular parallelograms have; without making any use of the phrase *ratio compounded of ratios*; and for a proper explanation of this term the reader may consult *Simson's* edition. I would propose it to the reader to examine these propositions first as problems, because this will fix his attention more to the particular steps of the construction and the consequences drawn from them. And also the extent and importance of these very beautiful theorems is more to be seen by considering them in this practical point of view. For by the twentieth proposition, figures may be increased or diminished in any proportion. For instance if I want to make a figure the fifth part of a given figure; I take the fifth part of one of the sides of the given figure; and find a mean

mean proportional between the line and its fifth part, upon which if a similar and similarly situated figure be described, it will be the fifth part of the given figure. Likewise by this proposition the forty seventh of the first book, may be extended to any similar rectilineal figures.

The twenty third is often quoted to prove, that, if there be four lines which have the same ratio; and also other four, the rectangles contained by the antecedents and consequents will have the same ratio, the first to the second which the third has to the fourth. This the reader should examine, which he may easily do by the assistance of the note below. *

The subject and arrangement of what remains is so obvious, that it would be paying the reader but a poor complement to suppose that he stood in need of any farther information, especially if he read what *Simson* has said upon the 28, and 29 propositions.

I have said nothing of corollaries; and it is doubtful in what sense Euclid would have them be understood. In the first four books, they are all some circumstances which are in fact demonstrated in the proposition, but which, not being exprest in words, the reader might without this notice have overlooked; but he seems afterwards not to confine himself to this sense, as will be obvious to the reader.

* See prop. 10. B. 1. Ham. Con. Sect.

$$\left.\begin{array}{l}\text{EP. PD. PF. PH}\\ \text{BM. MA. BM. MC.}\end{array}\right\}\begin{array}{l}\text{K. L. M.}\\ \text{N. O. P.}\end{array}$$

Then (by equal.) K : M :: N : P.

THE

ELEMENTS

OF

EUCLID.

BOOK V.

DEFINITIONS.

1. A Part is a magnitude of a magnitude, the less of the greater, when it measures the greater. 2. But a multiple, the greater of the less, when it is measured by the less. Book V.

3. Ratio is a certain habitude of magnitudes of the same kind to one another, which *is only* as to quantity.

4. Magnitudes are said to have a ratio to one another, which may be multiplied so as to exceed one another.

5. Magnitudes are said to be in the same ratio; the first to the second and the third to the fourth; when the equimultiples of the first and third, according to any multiplication, are at the same time less, or at the same time equal, or are at the same time greater than each of the equimultiples of the second and fourth, compared with one another. 6. And let magnitudes, having the same ratio, be called proportionals. 7. But when of equimultiples; the multiple of the

Book V. the firſt exceeds the multiple of the ſecond, but the multiple of the third does not exceed the multiple of the fourth; then the firſt is ſaid to have to the ſecond a greater ratio than the third to the fourth. 8. And proportion is a ſimilitude of ratios. 9. But proportion conſiſts in three terms at leaſt. 10. And when three magnitudes are proportionals; the firſt is ſaid to have to the third a duplicate ratio of that which it has to the ſecond. 11. And when four magnitudes are proportionals; the firſt is ſaid to have to the fourth a triplicate ratio of that which it has to the ſecond; and always in order one more, as long as the proportion continues.

12. The leading magnitudes are ſaid to be of like ratio with the leading magnitudes; and thoſe that follow *are ſaid to be of like ratio* with thoſe that follow. Or. The antecedents are ſaid to be homologous magnitudes to the antecedents; and the conſequents to the conſequents.

13. Alternate ratio is the taking of the antecedent to the antecedent; and of the conſequent to the conſequent.

14. Inverſe ratio is the taking of the conſequent as an antecedent to the antecedent as a conſequent.

15. The compoſition of a ratio is the taking of the antecedent together with the conſequent, as one, to the conſequent. 16. But the diviſion of a ratio is the taking of the exceſs, by which the antecedent exceeds the conſequent, to the conſequent itſelf.

17. The converſion of a ratio is the taking of the antecedent, to the exceſs, by which the antecedent exceeds the conſequent.

18, A ratio of equality is, there being ſeveral magnitudes, and others equal to them in multitude, and in the ſame proportion taken two by two, when it is, as in the firſt magnitudes; the firſt to the laſt, ſo in the ſecond magnitudes the firſt to the laſt. or otherwiſe. The taking of the extremes by a ſubſtraction of the middle *terms*.

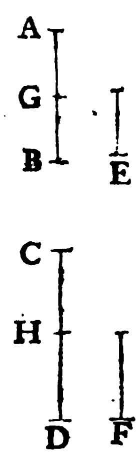
19. Ordinate proportion is, when it is as the antecedent to the conſequent, ſo is the antecedent to the conſequent; and it is as the conſequent to ſome other, ſo is the conſequent to ſome other. 20. But perturbate proportion is, when there being three magnitudes and others equal to them in number, it is as in the firſt magnitudes the antecedent to the conſequent, ſo in the ſecond magnitudes

tudes the antecedent to the conſequent; but as in the firſt magnitudes the conſequent is to ſome other, ſo in the ſecond magnitudes is ſome other to the antecedent. Book V.

PROP. I.

If there be any number of magnitudes, equimultiples of an equal number of magnitudes, each of each; whatſoever multiple one of the magnitudes is of one, the ſame multiple will all be of all.

Let AB, CD be any number of magnitudes, equimultiples of an equal number of magnitudes E, F; each of each: I ſay that whatſoever multiple AB is of E, the ſame multiple will AB, CD *together* be of E, F *together*.



For becauſe AB is the ſame multiple of E that CD is of F; as many magnitudes as there are in AB equal to E, ſo many are there in CD equal to F; let AB be divided (by 3. 1.) into the magnitudes AG, GB equal to E; and CD into the *magnitudes* CH, HD equal to F: certainly the number of *parts* CH, HD will be equal to the number of *parts* AG, GB: and becauſe AG is equal to E, and CH *equal* to F, therefore AG, CH *together are* equal to E and F *together*; for the ſame reaſon GB is equal to E, and HD to F; therefore alſo GB, HD *together* are equal to E, F *together*: therefore as many *magnitudes* as there are in AB equal to E; ſo many are there in AB, CD *together* equal to E, F *together*: wherefore whatſoever multiple AB is of E, the ſame multiple will AB, CD *together* be of E, F *together*.

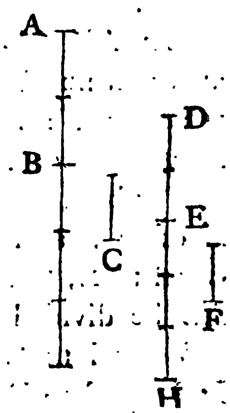
Wherefore if there be any number of magnitudes, equimultiples of an equal number of magnitudes, each of each; whatſoever multiple one of the magnitudes is of one, the ſame multiple will all be of all. Which was to be demonſtrated.

PROP. II.

If the firſt be the ſame multiple of the ſecond as the third *is* of the fourth; and if the fifth be the ſame multiple of the ſecond as the

Book V. the ſixth *is* of the fourth: alſo the firſt and fifth together will be the ſame multiple of the ſecond, as the third and ſixth together *is* of the fourth.

For let the firſt AB be the ſame multiple of the ſecond C as the third DE is of the fourth F; and let the fifth BG *be* the ſame multiple of the ſecond C as the ſixth EH *is* of the fourth F; I ſay that AG the firſt together with the fifth will be the ſame multiple of the ſecond C as DH the third and ſixth *is* of F the fourth.



For becauſe AB is the ſame multiple of C that DE *is* of F; as many magnitudes as there are in AB equal to C, ſo many *will there be* in DE equal to F: certainly for the ſame reaſon alſo, as many *magnitudes* as there are in BG equal to C, ſo many are there alſo in EH equal to F; therefore as many as there are in the whole AG equal to C ſo many *are there* in the whole DH equal to F: wherefore whatſoever multiple AG is of C, the ſame multiple will DH be of F; therefore AG the firſt together with the fifth will be the ſame multiple of C the ſecond, that DA the third and ſixth *is* of F the fourth.

Wherefore if the firſt be the ſame multiple of the ſecond as the third *is* of the fourth; and if the fifth be the ſame multiple of the ſecond as the ſixth *is* of the fourth; alſo the firſt and fifth together will be the ſame multiple of the ſecond, as the third and ſixth together *is* of the fourth. Which was to be demonſtrated.

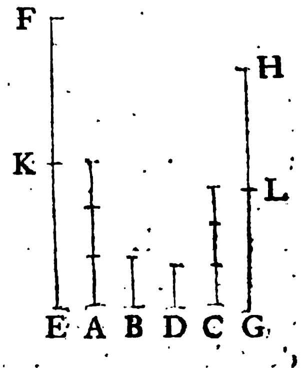
PROP. III.

If the firſt be the ſame multiple of the ſecond as the third *is* of the fourth; and equimultiples be taken of the firſt and third; alſo by equality, each of thoſe taken will be an equimultiple of each, the one of the ſecond, and the other of the fourth.

For let the firſt A be the ſame multiple of the ſecond B, as the third C *is* of the fourth D; and let EF, GH be taken equimultiples of A *and* C: I ſay that EF is the ſame multiple of B, which GH *is* of D.

For

For becauſe EF is the ſame multiple of A (by conſt.) which GH *is* of C: therefore as many *magnitudes* as there are in EF equal to A, ſo many *are there* in GH equal to C: let EF be divided into magnitudes equal to A, viz. EK, KF; and GH into *magnitudes* equal to C, viz. GL, LH; the number of *parts* EK, KF will be equal to the number of *parts* GL, LH: And becauſe A is the ſame multiple of B, as C *is* of D; and EK is equal to A, and GL equal to C; therefore EK is the ſame multiple of B, which GL is of D. Certainly for the ſame reaſon alſo KF is the ſame multiple of B, which LH is of D: wherefore becauſe the firſt EK is the ſame multiple of the ſecond B, as the third GL *is* of the fourth D; and alſo the fifth KF is the ſame multiple of the ſecond B, which the ſixth LH *is* of the fourth D; therefore (by 2. 5.) EF the firſt together with the fifth is the ſame multiple of the ſecond B, which GH the third and ſixth *is* of the fourth D.



Wherefore if the firſt be the ſame multiple of the ſecond, as the third is of the fourth; and equimultiples be taken of the firſt and third: alſo by equality, each of thoſe taken will be an equimultiple of each, the one of the ſecond, and the other of the fourth. Which was to be demonſtrated.

PROP. IV.

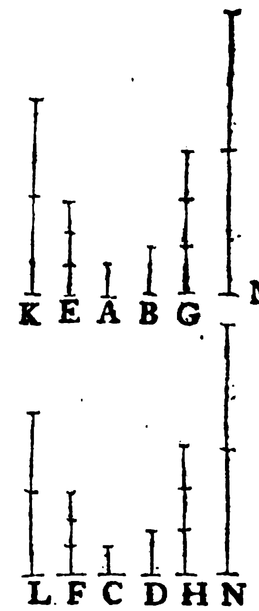
If the firſt have the ſame ratio to the ſecond as the third to the fourth; alſo the equimultiples of the firſt and third, will have the ſame ratio to the equimultiples of the ſecond and fourth, according to any multiplication whatſoever; taken ſo as to anſwer each other.

For let the firſt A have the ſame ratio to the ſecond B, which the third C *has* to the fourth D; and let E *and* F be taken equimultiples of A *and* C; but let G and H *be taken* any equimultiples of B and D which may accidentally happen; I ſay that as E is to G ſo is F to H.

For let K *and* L be taken equimultiples of E *and* F; and M, N *any other* equimultiples of G *and* H *which may happen*.

And

Book V. And becauſe E is the ſame multiple of A which F is of C; and K *and* L have been taken equimultiples of E *and* F; therefore (by 3. 5.) K is the ſame multiple of A which L *is* of C; Certainly for the ſame reaſon M is the ſame multiple of B which N *is* of D: and becauſe (by ſupp.) A is to B as C is to D; and K *and* L have been taken equimultiples of A *and* C; and M *and* N [any other] equimultiples of B *and* D [which may accidentally happen]; therefore (by 5 def. 5.) if K exceed M alſo L exceeds N; and if equal, equal; and if leſs, leſs: and K *and* L are equimultiples of E *and* F; and M *and* N any other equimultiples, which may accidentally happen; of G *and* H; therefore (by 5 def. 5.) E is to G as F is to H.



Wherefore if the firſt have the ſame ratio to the ſecond as the third to the fourth; alſo the equimultiples of the firſt and third, will have the ſame ratio to the equimultiples of the ſecond and fourth; according to any multiplication whatſoever; taken ſo as to anſwer each other. Which was to be demonſtrated.

Cor. Wherefore becauſe it has been demonſtrated, that if K exceed M; L exceeds N, and if equal, equal: and if leſs, leſs. Certainly alſo if M exceed K, N exceeds L: and if equal, equal; and if leſs, leſs: and on this account it will be; as G is to E, ſo is H to F. Certainly from this it is manifeſt that if four magnitudes be proportionals, they will be inverſly proportionals; (by def. 14. 5.).

PROP. V.

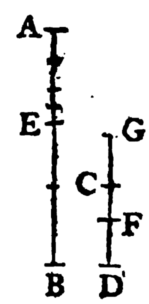
If a magnitude be the ſame multiple of a magnitude, which a *magnitude* taken from *the one* is of *a magnitude* taken from *the other*; whatſoever multiple the whole is of the whole, alſo the remainder will be the ſame multiple of the remainder.

For let the magnitude AB be the ſame multiple of the magnitude CD, which AE the *magnitude* taken from *the one* is of the *magnitude*

magnitude CF taken from *the other*: I say that whatsoever multiple the whole AB is of the whole CD, also EB the remainder will be the same multiple of FD the remainder, Book V.

For whatsoever multiple AE is of CF; let EB be the same multiple of CG.

And because AE is the same multiple of CF (by the const. and 1. 5.) *which* AB is of GF; and AE is supposed *to be* the same multiple of CF *which* AB is of CD: therefore AB is the same multiple of either of the *lines* GF, CD; therefore (by com. not. 7.) GF is equal to CD; let CF *which is* common be taken away; therefore the remainder GC is equal to the remainder DF; and because AE is the same multiple of CF which EB is of GC; and GC is equal to DF; therefore AE is the same multiple of CF which EB is of FD; but AE is supposed *to be* the same multiple of CF, which AB is of CD: therefore EB is the same multiple of FD *which* AB is of CD; therefore whatsoever multiple the whole AB is of the whole CD; also the remainder EB is the same multiple of the remainder FD.



Wherefore if a magnitude be the same multiple of a magnitude, which a *magnitude* taken from *the one* is of a magnitude taken from *the other*; whatsoever multiple the whole is of the whole, also the remainder will be the same multiple of the remainder. Which was to be demonstrated.

PROP. VI.

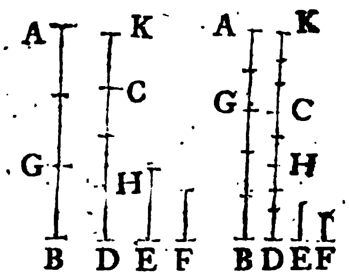
If two magnitudes be equimultiples of two magnitudes; and some *magnitudes* equimultiples of the same be taken away; also the remainders are either equal to those magnitudes, or equimultiples of them.

For let the two magnitudes AB, CD be equimultiples of the two magnitudes E, F; and let AG, CH the *magnitudes* taken away be equimultiples of the same E, F: I say that the remainders GB, HD are either equal to E, F or equimultiples of them.

For first let GB be equal to E; I say that HD is also equal to F: for put CK equal to F.

And

Book V. And becauſe AG is the ſame multiple of E as CH *is* of F; and GB *is* equal to E; and CK *equal* to F; therefore (by 2. 5.) AB is the ſame multiple of E *which* KH *is* of F; but AB is ſuppoſed *to be* the ſame multiple of E, *which* CD *is* of F; therefore KH is the ſame multiple of F, which CD *is* of F; wherefore becauſe each of the *lines* KH, CD *is* the ſame multiple of F; therefore (by com. not. 6.) KH is equal to CD; let CH *which is* common be taken away; therefore the remainder KC is equal to the remainder HD: but KC is equal to F; therefore HD is alſo equal to F; ſo that when GB is equal to E, HD will alſo be equal to F.



Certainly in the ſame manner we ſhall demonſtrate, that when GB is a multiple of E, that HD will be the ſame multiple of F.

Wherefore if two magnitudes be equimultiples of two magnitudes; and ſome *magnitudes* equimultiples of the ſame be taken away; alſo the remainders are either equal to thoſe magnitudes, or equimultiples of them. Which was to be demonſtrated.

PROP. VII.

Equal *magnitudes* have the ſame ratio to the ſame *magnitude*; and the ſame *magnitude has the ſame ratio* to equal *magnitudes*.

Let A *and* B be equal magnitudes; and C any other magnitude which may accidentally happen; I ſay that each of the magnitudes A, B has the ſame ratio to C; and that C has the ſame ratio to each of the *magnitudes* A, B.

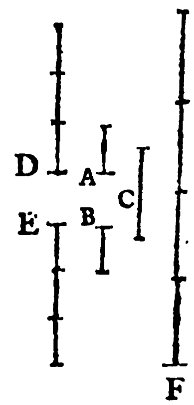
For let D, E be taken equimultiples of A, B; and let F be taken any other multiple of C which may accidentally happen.

And becauſe D is the ſame multiple of A, *which* E *is* of B; and A *is* (by ſupp.) equal to B; therefore D *is* equal to E; but F *is* any other multiple of C which may accidentally happen; wherefore if D exceed F, E alſo exceeds F; and if equal, equal; and if leſs, leſs: and D, E are equimultiples of A, B; and F any other multiple of C which may accidentally happen; therefore (by

(by 5 def. 5.) it is as A *is* to C ſo is B to C.

I ſay alſo that C has the ſame ratio to each of the *magnitudes* A, B,

For the ſame things being conſtructed; in the ſame manner we ſhall demonſtrate that D is equal to E; and that F is any other *magnitude*; therefore if F exceed D; it alſo exceeds E: and if equal, equal: and if leſs, leſs: and F is a multiple of C; and D, E any other equimultiples of A and B which may accidentally happen; therefore (by 5. def. 5.) it is, as C to A ſo is C to B.



Therefore equal *magnitudes* have the ſame ratio to the ſame *magnitude*; and the ſame *magnitude has the ſame ratio* to equal *magnitudes*. Which was to be demonſtrated.

PROP. VIII.

Of unequal magnitudes; the greater has a greater ratio to the ſame *magnitude*, than the leſs *has*: and the ſame *magnitude* has a greater ratio to the leſs, than it has to the greater.

Let AB and C be unequal magnitudes; and let AB be greater than C; and let D be any other magnitude which may accidentally happen: I ſay that AB has a greater ratio to D than C has to D: And D has a greater ratio to C, than *it has* to AB.

For becauſe AB is greater than C; make BE equal to C: certainly (by 4. def. 5.) the leſſer of the *two* AE, EB being multiplied will at length be greater than D. Firſt let AE be leſs than EB; and let AE be multiplied until what is produced ſhall be greater than D: and let FG be the multiple of it, which is greater than D: and whatſoever multiple FG is of AE, let GH be the ſame multiple of EB; and K *the ſame multiple* of C: and let L be taken the double of D, and M triple; and more by one in order, until the multiple of D taken becomes the firſt greater than K: let it be taken: and let it be N four times D, the firſt greater than K.

Wherefore becauſe K is the firſt leſs than N *or* [*N is the firſt multiple of D greater than K*]; therefore K is not leſs than M;

Book V. and becaufe FG is the fame multiple of AB which GH *is* of EB; therefore (by 1. 5.) FG is the fame multiple of AE which FH is of AB; and FG is the fame multiple of AB which K is of C; therefore FH is the fame multiple of AB, which K is of C; therefore FH and K are equimultiples of AB and C: Again becaufe GH is the fame multiple of EB which K *is* of C; and EB is equal to C; therefore GH is equal to K: but K is not lefs than M; therefore neither is GH lefs than M; but (by conft.) FG is greater than D; therefore the whole FH is greater than both D and M together; but D and M together are equal to N; therefore FH exceeds N, but K does not exceed N; and FH and K are equimultiples of AB and C; and N is any other multiple of D which may accidentally happen; therefore (by 7. def. 5.) AB has a greater ratio to D, than C has to D.

I fay alfo, that D has a greater ratio to C, than D *has* to AB.

For the fame things being conftructed; in like manner we fhall demonftrate, that N exceeds K; but does not exceed FH; and N is a multiple of D; and FH *and* K [any other] equimultiples of AB and C [which may accidentally happen]; therefore (by 7. def. 5.) D has a greater ratio to C, than D *has* to AB.

But let AE be greater than EB; certainly EB the lefs being multiplied will at length (by 4. def. 5.) be greater than D: let it be multiplied; and let GH be the multiple of EB, greater than D; and whatfoever multiple GH is of EB; let FG be made the fame multiple of AE; and K of C: Certainly in the fame manner we fhall demonftrate, that FH *and* K are equimultiples of AB *and* C: and in like manner let N be taken, a multiple of D, the firft greater than FG; fo that again FG be not lefs than M; but GH is greater than D; therefore the whole FH exceeds D and M together, that is N; but K does not exceed N; fince FG being

being greater than GH; that is K, does not exceed N; and in like manner following the *ſteps* above we finiſh the demonſtration. Book V.

Wherefore of unequal magnitudes, the greater has a greater ratio to the ſame *magnitude*, than the leſs *has*: and the ſame *magnitude* has a greater ratio to the leſs, than it has to the greater. Which was to be demonſtrated.

PROP. IX.

Magnitudes having the ſame ratio to the ſame *magnitude* are equal to one another; and thoſe are alſo equal to one another, to which the ſame *magnitude* has the ſame ratio.

For let each of the *magnitudes* A, B have the ſame ratio to the ſame *magnitude* C: I ſay that A is equal to B.

For if not, (by 8. 5.) each of the *magnitudes* A, B could not have the ſame ratio to C: but it has (by ſupp.); therefore A is equal to B.


B
A
C

Again let C have the ſame ratio to each of the *magnitudes* A, B; I ſay that A is equal to B.

For if not; C could not (by 8. 5.) have the ſame ratio to each of the *magnitudes* A, B; but it has (by ſupp.); therefore A is equal to B.

Wherefore, *magnitudes* having the ſame ratio to the ſame *magnitude* are equal to one another; and thoſe are equal to one another to which the ſame *magnitude* has the ſame ratio. Which was to be demonſtrated.

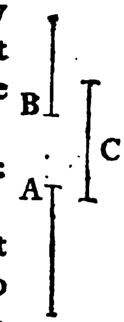
PROP. X.

Of *magnitudes* having ratio to the ſame *magnitude*, that is the greater which has the greater ratio: and that is the leſs to which the ſame has a greater ratio.

For let A have a greater ratio to C than B has to C: I ſay that A is greater than B.

For if not; either A *is* equal to B, or leſs; but A is not equal to B; for *then* each of the *magnitudes* A, B (by 7. 5.) has the ſame ratio to C; but (by ſupp.) *each of them* has not: therefore A is not equal to B; neither is A leſs than B; for (by 8. 5.) A would

Book V. have a lefs ratio to C than B *would have* to C; but (by fupp.) it has not; therefore A is not lefs than B; but it has been demonftrated that neither is it equal: therefore A is greater than B.



Again let C have a greater ratio to B than C has to A: I fay that B is lefs than A.

For if not; it is either equal, or greater: but B is not equal to A, for (by 7. 5,) C would have the fame ratio to each of the *magnitudes* A, B: but (by fupp.) it has not; therefore A is not equal to B: Neither is B greater than A; for (by 8. 5.) C would have a lefs ratio to B than to A; but (by fupp.) it has not; therefore B is not greater than A; but it has been demonftrated that neither *is* it equal; therefore B is lefs than A.

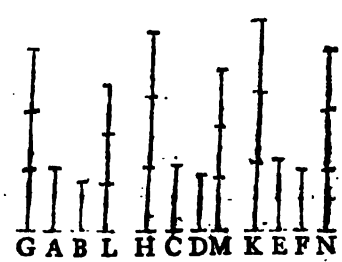
Wherefore of *magnitudes* having ratio to the fame *magnitude*, that is the greater, which has the greater ratio: and that is the lefs to which the fame has a greater ratio. Which was to be demonftrated.

PROP. XI.

Ratios which are the fame to the fame ratio, are the fame to one another.

For let the *ratios* be; as A to B fo is C to D; and as C to D fo is E to F; I fay that it is, as A is to B fo is E to F.

For let G, H, K be taken equimultiples of A, C, E; and let L, M, N *be taken* any other equimultiples of B, D, F which may accidentally happen.



And becaufe it is (by fupp.) as A *is* to B fo *is* C to D; and G *and* H have been taken equimultiples of A *and* C; and L, M any equimultiples of B, D which may accidentally happen; wherefore if G exceed L; H alfo exceeds M (by 5. def. 5.); and if equal, equal; and if lefs, lefs: Again, becaufe it is (by fupp.), as C *is* to D fo is E to F; and H, K have been taken equimultiples of C, E; and M, N any other equimultiples of D, F which may accidentally happen; wherefore (by 5. def. 5.) if H exceed M; K alfo exceeds N; aud if equal, equal;

equal; and if lefs, lefs; but if H exceed M; G alfo exceeds L;
and if equal, equal; and if lefs, lefs: and G, K are equimultiples of A, E; and L, N any other equimultiples of B, F which may accidentally happen; wherefore (by 5. def. 5.) it is, as A *is* to B fo *is* E to F.

Wherefore *ratios* which *are* the fame to the fame ratio; are the fame to one another. Which was to be demonftrated.

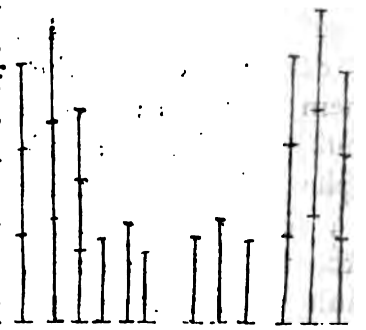
PROP. XII.

If any number of magnitudes be proportionals; it will be, as one of the antecedents *is* to one of the confequents, fo *are* all the antecedents to all the confequents.

Let A, B, C; D, E, F be any number of magnitudes, proportionals; viz. as A *is* to B fo *let* C *be* to D; and E to F: I fay that it is as A *is* to B fo is A, C, E to B, D, F.

For let G, H, K equimultiples of A, C, E be taken; and L, M, N any other which may accidentally happen of B, D, F.

And becaufe it is, as A *is* to B fo is C to D; and E to F (by fupp.); and G, H, K have been taken equimultiples of A, C, E; and L, M, N any other equimultiples of B, D, F which may accidentally happen; wherefore (by 5. def. 5.) if G exceed L; H alfo exceeds M and K *exceeds* N; and if equal, equal; and if lefs, lefs: fo that alfo if G exceed L; G, H, K *taken together* alfo exceeds



L, M, N *taken together*; and if *it be* equal; *they are* equal; and if *it be* lefs, *they are* lefs; and G; and G, H, K are equimultiples of A, and of A, C, E; fince (by 1. 5.) if there be any nnmber of magnitudes equimultiples of an equal number of magnitudes, each of each; whatfoever multiple any one magnitude is of one; the fame multiple will, all be of all: Certainly for the fame reafon alfo; L, and L, M, N are equimultiples of B, and of B, D, F: therefore (by 5. def. 5.) it is, as A *is* to B fo *is* A, C, E to B, D, F.

Wherefore if any number of magnitudes be proportionals; it will be as one of the antecedents *is* to one of the confequents, fo are

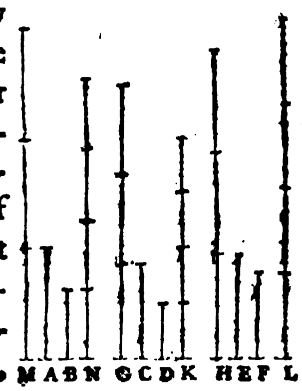
Book V. are all the antecedents to all the consequents. Which was to be demonstrated.

PROP. XIII.

If the first have the same ratio to the second which the third has to the fourth: but if the third have a greater ratio to the fourth than the fifth *has* to the fixth; the first also will have a greater ratio to the second than the fifth *has* to the sixth.

For let the first A have the same ratio to the second B, *which* the third C has to the fourth D: but let the third C have a greater ratio to the fourth D, than the fifth E, has to the sixth F; I say that the first A will have a greater ratio to the second B, than the fifth E has to the sixth F.

For because C has a greater ratio to D (by supp.), than E has to F; (by 7. def. 5.) there are some equimultiples of C and E; and other equimultiples [which may accidentally happen] of D and F: and the multiple of C exceeds the multiple of D: but the multiple of E does not exceed the multiple of F: let them be taken; and let G, H be the equimultiples of C, E; and K, L other equimultiples [which may accidentally happen] of D, F; so that G exceeds K; but H does not exceed L: and whatsoever multiple G is of C, let M be the same multiple of A: and whatsoever multiple K is of D, let N be the same multiple of B.



And because it is, as A *is* to B so *is* C to D; and M *and* G have been taken equimultiples of A, C; and N, K other equimultiples [which may accidentally happen] of B, D: wherefore (by 5. def. 5.) if M exceed N, G also exceeds K: and if equal, equal: and if less, less: But G exceeds K (by supp.) therefore also M exceeds N: but H does not exceed L; and M, H are equimultiples of A, E; and N, L other equimultiples [which may accidentally happen] of B, F; therefore (by 7. def. 5.) A has a greater ratio to B, than E *has* to F,

Where-

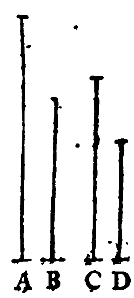
Wherefore if the first have the same ratio to the second which the third has to the fourth; but *if* the third have a greater ratio to the fourth than the fifth *has* to the sixth; the first also will have a greater ratio to the second than the fifth *has* to the sixth. Which was to be demonstrated.

PROP. XIV.

If the first have the same ratio to the second, *which* the third *has* to the fourth; and if the first be greater than the third; also the second will be greater than the fourth; and if equal, equal: and if less, less.

For let the first A have the same ratio to the second B, *which* the third C *has* to the fourth D; and let A be greater than C: I say that B is greater than D.

For because (by supp.) A is greater than C; and B is any other magnitude which may happen: therefore (by 8. 5.) A has a greater ratio to B, than C *has* to B; but as A *is* to B (by supp.) so *is* C to D; therefore (by 13. 5.) C has a greater ratio to D, than C *has* to B: but (by 10. 5.) that *magnitude* is the less, to which the same has a greater ratio; therefore D is less than B; so that B is greater than D.



Certainly in the same manner we shall demonstrate, than when A is equal to C, B will also be equal to D: and if A be less than C, B will be less than D.

Wherefore if the first have the same ratio to the second, which the third *has* to the fourth; and if the first be greater than the third; also the second will be greater than the fourth; and if equal, equal; and if less, less. Which was to be demonstrated.

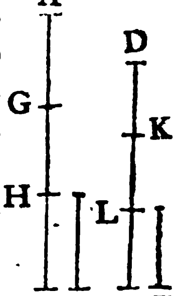
PROP. XV.

The parts, compared with one another, have the same ratio *which* their equimultiples *have*.

For let AB be the same multiple of C *which* DE *is* of F: I say that it is, as C *is* to F so *is* AB to DE.

For

For becauſe (by ſupp.) AB is the ſame multiple of C *which* DE *is* of F; therefore as many magnitudes as there are in AB equal to C, ſo many *is there* alſo in DE equal to F; let AB be divided into magnitudes equal to C, viz. AG, GH, HB; and DE into *magnitudes* equal to F, viz. DK, KL, LE: certainly the number of *parts* AG, GH, HB will be equal to the number of *parts* DK, KL, LE: and becauſe AG, GH, HB are equal to one another; alſo DK, KL, LE are equal to one another; therefore it is, (by 7. 5.) as AG *is* to DK ſo *is* GH to KL, and HB to LE; wherefore (by 12. 5.) it will be as one of the antecedents *is* to one of the conſequents, ſo *are* all the antecedents to all the conſequents; wherefore it is, as AG *is* to DK ſo *is* AB to DE; but AG *is* equal to C, and DK *is equal* to F; therefore it is, as C *is* to F ſo *is* AB to DE.



Wherefore the parts, compared with one another, have the ſame ratio *which* their equimultiples *have*. Which was to be demonſtrated.

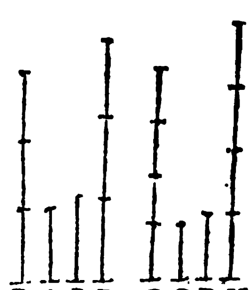
PROP. XVI.

If four magnitudes be proportionals; they will alſo be alternately proportionals.

Let the four magnitudes, A, B, C, D be proportionals; viz. as A *is* to B ſo *let* C *be* to D: I ſay that they will be alternately proportionals: viz. as A is to C ſo *will* B *be* to D.

For let E, F be taken equimultiples of A, B; and G, H any other equimultiples of C, D which may accidentally happen.

And becauſe E *is* the ſame multiple of A *which* F *is* of B; and (by 15. 5.) parts, compared with one another have the ſame ratio *which* their equimultiples *have*: therefore it is, as A *is* to B ſo *is* E to F; but (by ſupp.) as A *is* to B ſo *is* C to D; and therefore as C *is* to D ſo (by 11. 5.) is E to F: Again, becauſe G *and* H are equimultiples of C *and* D; therefore (by 15. 5.) C *is* to D as G *is* to H; but as C is to D ſo is E to



F;

F; therefore (by 11. 5.) as E is to F ſo is G to H; but if four magnitudes be proportionals, and the firſt be greater than the third; the ſecond (by 14. 5.) will be greater than the fourth: and if equal, equal: and if leſs, leſs: wherefore if E exceed G, F alſo exceeds H: and if equal, equal: and if leſs, leſs: and E and F are equimultiples of A *and* B, and G and H any other equimultiples of C *and* D which may accidentally happen; therefore (by 5. def. 5.) as A *is* to C ſo *is* B to D. Book V.

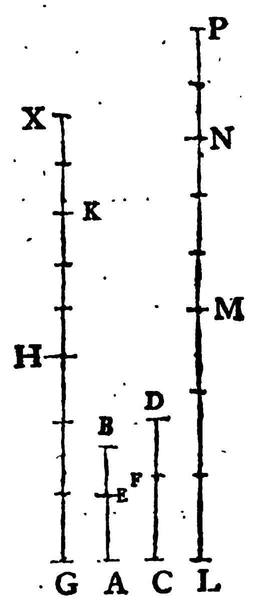
Wherefore if four magnitudes be proportionals they will be (by 13. def. 5.) alternately proportionals. Which was to be demonſtrated.

PROP. XVII.

If compounded magnitudes be proportionals; they will alſo be proportionals *when* divided.

Let the compounded magnitudes, AB, BE, CD, DF be proportionals; viz. as AB is to BE ſo *let* CD *be* to DF: I ſay that they will be proportionals when divided; that is (by 16. def. 5.) as AE *is* to EB ſo *will* CF *be* to FD.

For let GH, HK, LM, MN be taken equimultiples of AE, EB, CF, FD: and *let* KX, NP *be taken* any other equimultiples of EB, FD which may accidentally happen.



And becauſe GH is the ſame multiple of AE *which* HK *is* of EB; therefore GH is the ſame multiple of AE *which* GK *is* of AB (by 1. 5.); but GH is the ſame multiple of AE *which* LM *is* of CF: therefore GK is the ſame multiple of AB *which* LM *is* of CF. Again becauſe LM is the ſame multiple of CF *which* MN *is* of FD: therefore (by 1. 5.) LM is the ſame multiple of CF *which* LN *is* of CD: but LM was the ſame multiple of CF *which* GK *is* of AB: therefore GK is the ſame multiple of AB *which* LN *is* of CD: therefore GK, LN are equimultiples of AB, CD. Again becauſe HK is the ſame multiple of EB *which* MN *is* of FD; and KX is alſo the ſame multiple of EB *which* NP *is* of FD: alſo (by 2. 5.) HX is the ſame multiple of EB *which*

Book V. MP *is* of FD: and because it is (by supp.) as AB *is* to BE so *is* CD to DF; and GK, LN have been taken equimultiples of AB, CD; and HX, MP [any other] equimultiples of EB, FD [which may accidentally happen]; therefore (by 5. def. 5.) if GK exceed HX, LN also exceeds MP: and if equal, equal: and if less, less: Let GK exceed HX, and, the common *part* HK being taken away, therefore also GH exceeds KX: But if GK exceed HX, LN also exceeds MP; therefore also LN exceeds MP; and MN *which is* common being taken away, LM also exceeds NP: so that if GH exceed KX; LM also exceeds NP. Certainly in the same manner we shall demonstrate that if GH be equal to KX; LM will also be equal to NP: and if less, less: But GH, LM are equimultiples of AE, CF; and KX, NP any other equimultiples of EB, FD which may accidentally happen; therefore (by 5. def. 5.) it is, as AE *is* to EB so *is* CF to FD.

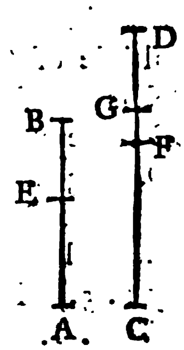
Wherefore if compounded magnitudes be proportionals; they will also be proportionals *when* divided. Which was to be demonstrated.

PROP. XVIII.

If magnitudes *when* divided are proportionals; they will be proportionals *when* compounded.

Let, AE, EB, CF, FD the divided magnitudes be proportionals, viz. as AE *is* to EB so *let* CF *be* to FD: I say also that they will be proportionals *when* compounded, *that is* (by 15. def. 5.) as AB *is* to BE so *will* CD *be* to DF.

For if it is not, as AB is to BE so *is* CD to DF; it will be, as AB *is* to BE so *is* CD, either to some *magnitude* less than FD, or to *one* greater.



Let it be first to DG a less: and because it is (by this supp.) as AB *is* to BE so is CD to DG, the compounded magnitudes are proportionals: so that also (by 17. 5.) they will be proportionals when divided; therefore it is, as AE is to EB so *is* CG to GD: but it is also supposed that as AE is to EB so is CF to FD: wherefore (by 11. 5.) as CG *is* to GD so *is* CF to FD: but CG the first *is* greater than CF the third; therefore also (by 14. 5.) the second GD is greater

than the fourth FD; but it is also less: which is impossible: Book V.
wherefore it is not, as AB *is* to BE so *is* CD to a *magnitude* less than DF: Certainly in the same manner we shall demonstrate, that neither *is* it to *one* greater. Therefore, it is to *that one* itself.

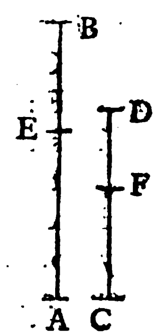
Wherefore if magnitudes *when* divided are proportionals; they will be proportionals *when* compounded. Which was to be demonstrated.

PROP. XIX.

If it be, as the whole *magnitude is* to the whole, so *is* a *magnitude* taken away to a *magnitude* taken away; the remainder also will be to the remainder as the whole *is* to the whole.

For let it be, as the whole AB *is* to the whole CD, so *let* AE a *magnitude* taken away *be* to CF a *magnitude* taken away; I say that EB the remainder will also be to FD the remainder, as the whole AB *is* to the whole CD.

For since it is, as the whole AB *is* to the whole CD so *is* AE to CF: also alternately (by 16. 5.) as AB *is* to AE so *is* CD to CF; and because the compounded magnitudes are proportionals, they will also be proportionals when divided (by 17. 5.); therefore as BE is to EA so is DF to FC; and therefore alternately (by 16. 5.), as BE *is* to DF so *is* EA to FC: but as EA is to FC so is the whole AB supposed *to be* to the whole CD; therefore also (by 11. 5.) the remainder EB will be to the remainder FD as the whole AB is to the whole CD.



Wherefore if it be, as the whole *magnitude is* to the whole, so is a *magnitude* taken away to a *magnitude* taken away; the remainder also will be to the remainder as the whole *is* to the whole. Which was to be demonstrated.

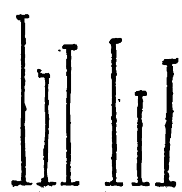
Cor. And because it has been demonstrated (in this prop.) as AB is to CD so *is* EB to FD: and alternately (by 16. 5.) as AB *is* to EB so *is* CD to FD: therefore the compounded magnitudes are proportionals: but it has been demonstrated (in this and prop. 16. 5.) as AB is to AE so is CD to CF; and *this* is the conversion *of the proportionals*. Certainly from this it is manifest, that if compounded

Book V. pounded magnitudes be proportionals, they will be proportionals by converſion (by 17. def. 5.). Which was to be demonſtrated.

PROP. XX.

If there be three magnitudes and others, equal to them in multitude, in the ſame ratio, taken two and two; and by equality, if the firſt be greater than the third; alſo the fourth will be greater than the ſixth: and if equal, equal: and if leſs, leſs.

Let A, B, C be three magnitudes, and D, E, F others equal to them in multitude, in the ſame ratio taken two and two; viz. as A is to B ſo *let* D *be* to E, and as B is to C ſo *let* E *be* to F: and by equality, let A be greater than C; I ſay alſo that D will be greater than F: and if equal, equal: and if leſs, leſs.



A B C D E F

For becauſe A is greater than C, and B *is* any *magnitude* which may accidentally happen: but (by 8. 5.) the greater has a greater ratio to the ſame than the leſs *has*: therefore A has a greater ratio to B than C *has* to B: but (by ſupp.) as A is to B ſo is D to E: but by inverſion (by cor. to 4. 5.) C is to B as F is to E: therefore (by 11. et 13. of 5.) D has a greater ratio to E than F has to E: but of *magnitudes* having ratio to the ſame; that is the greater which has the greater ratio (by 10. 5.): therefore D is greater than F: Certainly in the ſame manner we ſhall demonſtrate, that if A be equal to C, D will alſo be equal to F: and if leſs, leſs.

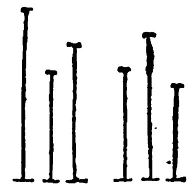
Wherefore if there be three magnitudes, and others equal to them in multitude, in the ſame ratio, taken two and two; and by equality, if the firſt be greater than the third; alſo the fourth will be greater than the ſixth; and if equal, equal: and if leſs, leſs. Which was to be demonſtrated.

PROP. XXI.

If there be three magnitudes, and others equal to them in multitude; and in the ſame ratio, taken two and two, and their proportion be perturbate; and if by equality the firſt be greater than the

the third, the fourth will also be greater than the sixth: and if equal, equal: and if less, less. Book V.

Let A, B, C be three magnitudes, and D, E, F others equal to them in multitude, and in the same ratio, taken two and two; and let their proportion be perturbate, viz. as A *is* to B so *let* E *be* to F; and as B *is* to C so *let* D *be* to E: and by equality let A be greater than C: I say also that D will be greater than F: and if equal, equal: and if less, less.



For because A is greater than C, and B is some other *magnitude*: therefore (by 8. 5.) A has a greater ratio to B than C *has* to B: but as A *is* to B so (by supp.) is E to F: and as C is to B so by inversion *is* E to D: therefore (by 11. and 13. 5.) E has a greater ratio to F than E has to D: but (by 10. 5.) that is the less *magnitude* to which the same has a greater ratio; therefore F is less than D: wherefore D is greater than F: Certainly we shall demonstrate in the same manner, that if equal, equal; certainly if A be equal to C, D also will be equal to F: and if less, less.

Wherefore if there be three magnitudes and others equal to them in multitude; and in the same ratio, taken two and two, and their proportion be perturbate; and *if* by equality the first be greater than the third, the fourth will also be greater than the sixth: and if equal, equal :and if less, less. Which was to be demonstrated.

PROP. XXII.

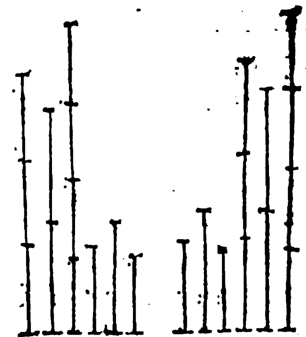
If there be any number of magnitudes; and others equal to them in multitude; in the same ratio, taken two and two;. also by equality they will be in the same ratio.

Let there be any number of magnitudes A, B, C; and D, E, F others equal to them in multitude; in the same ratio, taken two and two; viz. as A *is* to B so *let* D *be* to E and as B *is* to C so let E *be* to F: I say also, that by equality they will be in the same ratio; viz. (by 18. def. 5.) as A *is* to C so *will* D *be* to F.

For let G, H be taken equimultiples of A, D; and K, L any other equimultiples of B, E which may accidentally happen; and

besides

Book V. besides, M, N any other equimultiples of C, F which may accidentally happen.



And because it is (by supp.) as A *is* to B so *is* D to E; and G, H have been taken equimultiples of A, D; and K, L any other equimultiples of B, E which may accidentally happen: therefore (by 4. 5.) it is, as G *is* to K so *is* H to L. Certainly for the same reason also as K *is* to M so *is* L to N: wherefore because there are three magnitudes G, K, M; and others equal to them in multitude H, L, N and in the same ratio, taken two and two; therefore by equality, (by 20. 5.) if G exceed M, also H exceeds N: and if equal, equal: and if less, less: and G and H are equimultiples of A, D; and M, N any other equimultiples of C, F which may accidentally happen: therefore (by 5. def. 5.) it is, as A *is* to C so *is* D to F.

Wherefore if there be any number of magnitudes, and others equal to them in multitude; in the same ratio, taken two and two; also by equality, they will be in the same ratio. Which was to be demonstrated.

PROP. XXIII.

If there be three magnitudes, and others equal to them in multitude; in the same ratio, taken two and two; and *if* their proportion be perturbate; they will also be in the same ratio by equality.

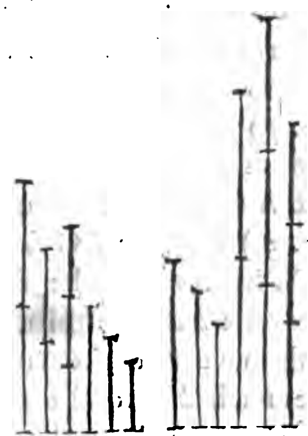
Let A, B, C be three magnitudes, and D, E, F others equal to them in multitude; in the same ratio, taken two and two; and let their proportion be perturbate, viz. as A *is* to B so *let* E *be* to F; and as B *is* to C so *let* D *be* to E: I say that it is, as A *is* to C so *is* D to F.

For let G, H, K be taken equimultiples of A, B, D; and L, M, N any other equimultiples of C, E, F which may accidentally happen.

And because G, K are equimultiples of A, B; and parts, (by 15. 5.) have the same ratio *to one another* as their equimultiples; therefore it is, as A *is* to B so *is* G to H: Certainly for the same reason also as E *is* to F so *is* M to N: and because it is, as B *is* to C so

is

is D to E; and H, K have been taken equimultiples of B, D; and L, M any other equimultiples of C, E which may accidentally happen; therefore it is, (by 4. 5.) as H *is* to L so *is* K to M: but it has been demonstrated as G *is* to H so *is* M to N: wherefore because there are three magnitudes G, H, L and others K, M, N equal to them in multitude; in the same ratio taken two and two; and their proportion is perturbate; therefore by equality (by 21. 5.) if G exceed L; K also exceeds N: and if equal, equal: and if less, less: and G, K are equimultiples of A, D; and L, N *any other* equimultiples of C, F *which may accidentally happen*; therefore (by 5. def. 5.) A *is* to C as D *is* to F.



Wherefore if there be three magnitudes, and others equal to them in multitude; in the same ratio taken two and two; and if their proportion be perturbate; they will also be in the same ratio by equality. Which was to be demonstrated.

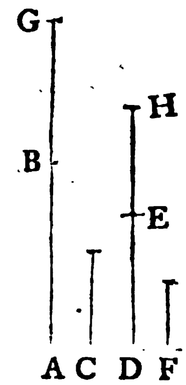
PROP. XXIV.

If the first have the same ratio to the second *which* the third *has* to the fourth; and if the fifth have also the same ratio to the second *which* the sixth *has* to the fourth; also the first and fifth together will have the same ratio to the second *which* the third and sixth *together has* to the fourth.

For let the first AB have the same ratio to the second C, *which* the third DE *has* to the fourth F: and also let the fifth BG have the same ratio to the second C, *which* the sixth EH *has* to the fourth F: I say that also AG the first together with the fifth, will have the same ratio to C the second *which* DH the third *together* with the sixth *has* to F the fourth.

For because it is, as BG *is* to C so is EH to F: therefore by inversion (by cor. to 4. 5.) C *is* to BG as F *is* to EH; Wherefore because it is as AB *is* to C so *is* DE to F; and as C *is* to BG so *is* F to

Book V. F to EH: therefore by equality (by 22. 5.) it is, as AB *is* to BG ſo *is* DE to EH: and becauſe the divided magnitudes are proportionals, they will alſo (by 18. 5) be proportionals *when* compounded: wherefore as AG is to GB ſo is DH to HE: but it is, as GB *is* to C ſo is HE to F: therefore by equality (by 22. 5.) it is, as AG *is* to C ſo *is* DH to F.



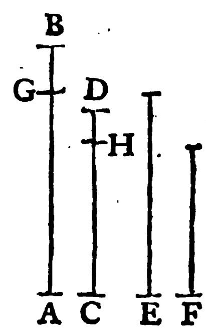
Wherefore if the firſt have the ſame ratio to the ſecond *which* the third *has* to the fourth; and *if* the fifth have alſo the ſame ratio to the ſecond *which* the ſixth *has* to the fourth; alſo the firſt and fifth together will have the ſame ratio to the ſecond *which* the third and ſixth together has to the fourth. Which was to be demonſtrated.

PROP. XXV.

If four magnitudes be proportionals; the greateſt and the leaſt of them are greater than the other two.

Let the four magnitudes AB, CD, E, F be proportionals; viz. as AB *is* to CD ſo *let* E *be* to F: and let AB be the greateſt of them and F the leaſt; I ſay that AB *and* F are greater than CD *and* E.

For let AG be made equal to E and *let* CH *be made* equal to F.



Wherefore becauſe it is, as AB *is* to CD ſo *is* E to F; and AG is equal to E and CH to F; therefore it is as AB is to CD ſo is AG to CH: and becauſe it is as the whole AB *is* to the whole CD ſo is AG taken away to CH taken away; therefore (by 19. 5.) the remainder GB will be to the remainder HD as the whole AB is to the whole CD: but (by ſupp.) AB is greater than CD; therefore GB is greater than HD: and becauſe AG is equal to E and CH to F: therefore AG and F *together* are equal to CH and E *together*; and becauſe if equals be added to unequals, the wholes are unequal: wherefore GB and HD being unequal: and AG and F be added to the greater GB: and CH and E be added to the *leſs* HD: AB and F together are greater than CD and E.

Wherefore if four magnitudes be proportionals, the greateſt and the leaſt of them are greater than the other two. Which was to be demonſtrated.

THE

ELEMENTS

OF

EUCLID.

BOOK VI.

DEFINITIONS.

1. SIMILAR rectilineal figures are those which have their angles equal each to each; and the sides about the equal angles proportionals. Book VI. 2. But those are reciprocal figures; when the antecedent and consequent terms are in each of the figures.

3. A straight line is said to have been cut in extreme and mean ratio, when it is, as the whole *line* is to the greater segment, so is the greater *segment* to the less.

4. The altitude of any figure is the perpendicular drawn from the vertex to the base.

5. A ratio is said to be compounded of ratios, when the quantities of the ratios being multiplied into one another do make some *ratio*.

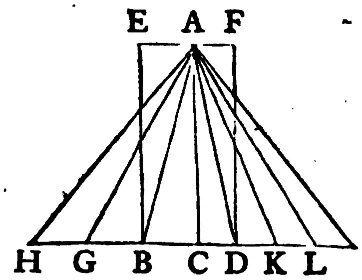
Book VI.

PROP. I.

The triangles and parallelograms which are under the ſame altitude are to one another as *their* baſes.

Let ABC, ACD be triangles; and EC, CF parallelograms, which are under the ſame altitude, viz. the perpendicular drawn from *the point* A to BD: I ſay that it is, as the baſe BC *is* to the baſe CD ſo *is* the triangle ABC to the triangle ACD: alſo ſo *is* the parallelogram EC to the parallelogram CF.

For let BD be produced towards both parts to the points H, L: and let any number of *lines* BG, GH be made equal to the baſe BC: and any number of *lines* DK, KL equal to the baſe CD: and let AG, AH, AK, AL be joined.



And becauſe CB, BG, GH are equal to one another; alſo (by 38. 1.) the triangles AGH, AGB, ABC are equal to one another: therefore whatſoever multiple the baſe HC is of the baſe BC; the ſame multiple alſo is the triangle AHC of the triangle ABC. Certainly for the ſame reaſon alſo, whatſoever multiple the baſe CL is of the baſe CD the ſame multiple is the triangle ACL of the triangle ACD. And if the baſe HC be equal to the baſe CL (by 38. 1.) the triangle AHC is alſo equal to the triangle ACL: and if the baſe HC exceed the baſe CL; the triangle AHC alſo exceeds the triangle ACL: and if leſs, leſs. There being four magnitudes, the two baſes BC, CD; and the two triangles ABC, ACD: *and* equimultiples of the baſe BC and of the triangle ABC have been taken viz. the baſe HC and the triangle AHC: and of the baſe CD and of the triangle ACD any other equimultiples which may accidentally happen viz. the baſe CL and the triangle ACL: and it has been demonſtrated that if the baſe HC exceed the base CL; the triangle AHC alſo exceeds the triangle ACL: and if equal, equal: and if leſs, leſs: therefore (by 5. def. 5.) it is, as, the baſe BC *is* to the baſe CD ſo *is* the triangle ABC to the triangle ACD.

And

And because the parallelogram EC is double of the triangle ABC (by 41. 1.): and the parallelogram FC is double of the triangle ACD; and parts (by 15. 5.) have the same ratio *to one another* as their equimultiples; therefore it is, as the triangle ABC is to the triangle ACD so *is* the parallelogram EC to the parallelogram FC: wherefore because it has been demonstrated *that* as the base BC *is* to the base CD so is the triangle ABC to the triangle ACD: and as the triangle ABC *is* to the triangle ACD so is the parallelogram EC to the parallelogram FC; therefore also (by 11. 5.) the base BC is to the base CD as the parallelogram EC *is* to the parallelogram FC.

Wherefore the triangles and parallelograms, which are under the same altitude, are to one another as *their* bases. Which was to be demonstrated.

PROP. II.

If any straight line be drawn parallel to one of the sides of a triangle, it will cut the sides of the triangle proportionally: and if the sides of the triangle be cut proportionally, the straight line joining the sections will be parallel to the remaining side of the triangle.

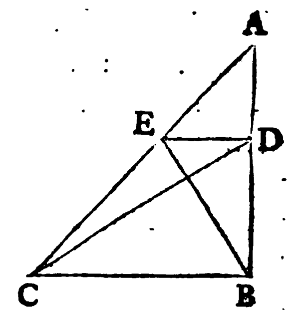
For let DE be drawn parallel to BC one of the sides of the triangle ABC: I say that it is, as BD *is* to DA so *is* CE to EA.

For let BE, CD be joined.

Therefore the triangle BDE is equal to the triangle CDE; for they are upon the same base DE and between the same parallel lines DE, BC: but ADE is some other triangle; and (by 7. 5.) equal *magnitudes* have the same ratio to the same *magnitude*: therefore it *is*, as the triangle BDE *is* to the triangle ADE so *is* the triangle CDE to the triangle ADE: but as the triangle BDE is to the triangle ADE so *is* (by 1. 6.) BD to DA: for being under the same altitude, the perpendicular drawn from the *point* E to AB, they are to one another as their bases. Certainly for the same reason also, as the triangle CDE *is* to the triangle ADE so *is* CE to EA: Therefore (by 11. 5.) as BD is to DA so is CE to EA.

But let AB, AC, the sides of the triangle ABC be cut proportionally in the points D, E viz. as BD *is* to DA so *let* CE *be* to EA:

Book VI. and let DE be joined: I ſay that DE is parallel to BC.



For the ſame things being conſtructed; becauſe it is, as BD *is* to DA ſo *is* CE to EA: but as BD *is* to DA ſo (by 1. 6.) *is* the triangle BDE to the triangle ADE: and as CE is to EA ſo is the triangle CDE to the triangle ADE: therefore alſo (by 11. 5.) as the triangle BDE *is* to the triangle ADE ſo *is* the triangle CDE to the triangle ADE: therefore each of the triangles BDE, CDE has the ſame ratio to the triangle ADE; therefore (by 9. 5.) the triangle BDE is equal to the triangle CDE: and they are upon the ſame baſe DE: but equal triangles being upon the ſame baſe are alſo (by 39. 1.) between the ſame parallels: therefore DE is parallel to BC.

Wherefore if any ſtraight line be drawn parallel to one of the ſides of a triangle, it will cut the ſides of the triangle proportionally; and if the ſides of the triangle be cut proportionally, the ſtraight line joining the ſections will be parallel to the remaining ſide of the triangle. Which was to be demonſtrated.

PROP. III.

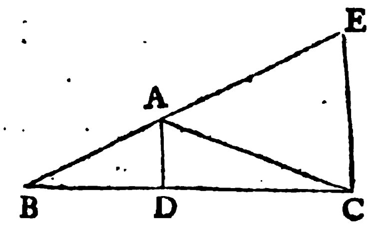
If an angle of a triangle be cut in halves, and the ſtraight line cutting the angle alſo cut the baſe; the ſegments of the baſe will have the ſame ratio *which* the remaining ſides of the triangle *have to one another*: and if the ſegments of the baſe have the ſame ratio *which* the remaining ſides of the triangle *have to one another*; the ſtraight line joining the vertex *and the point of* ſection cuts the angle of the triangle in halves.

Let ABC be a triangle, and let the angle BAC be cut in halves by the ſtraight line AD: I ſay that it is as BD *is* to DC ſo *is* BA to AC.

For let CE be drawn through the point C parallel to DA: and BA being produced let it meet *CE* in E.

And becauſe the ſtraight line AC hath fallen upon the parallels AD, EC; therefore the angle ACE (by 29. 1.) is equal to the angle

angle CAD: but the angle CAD is supposed equal to BAD: therefore the angle BAD is equal to the angle ACE. Again because the straight line BAE hath fallen upon the parallels AD, EC; the outward angle BAD is equal (by 29. 1.) to the inward *angle* AEC: but



the angle ACE hath been also demonstrated *to be* equal to BAD; therefore (by com. not. 1.) the *angle* ACE is equal to AEC: so that also (by 6. 1.) the side AE is equal to the side AC: and because AD has been drawn *parallel* to CE one of the sides of the triangle BCE; there is *this* proportion therefore (by 2. 6.) as BD is to DC so *is* BA to AE: But AE is equal to AC: therefore it is (by 7. 5.) as BD *is* to DC so *is* BA to AC.

But let BD be to DC as BA to AC; and let AD be joined: I say that the angle BAC hath been cut in halves by the straight line AD.

For the same things being constructed; because it is (by supp.), as BD *is* to DC so *is* BA to AC: but also (by 2. 6.) as BD *is* to DC so *is* BA to AE; for AD hath been drawn parallel to CE one of the sides of the triangle BCE: and therefore (by 11. 5.) as BA *is* to AC so *is* BA to AE; therefore (by 9. 5.) AC is equal to AE; so that also (by 5. 1.) the angle AEC is equal to the angle ACE: but the *angle* AEC is equal to the outward *angle* BAD (by 29. 1.); and also the *angle* ACE is equal to the alternate *angle* CAD; therefore the *angle* BAD is equal to CAD: wherefore the *angle* BAC hath been cut in halves by the straight line AD.

Wherefore if an angle of a triangle be cut in halves; and the straight line cutting the angle also cut the base; the segments of the base will have the same ratio *which* the remaining sides of the triangle *have to one another*: and if the segments of the base have the same ratio *which* the remaining sides of the triangle *have to one another*; the straight line, joining the vertex and the point of section, cuts the angle of the triangle in halves. Which was to be demonstrated.

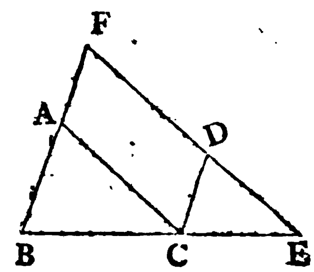
PROP.

PROP. IV.

The ſides about the equal angles of equiangular triangles are proportionals; and thoſe ſides are of like ratio *which are* extended under the equal angles.

Let ABC, DCE be equiangular triangles, having the angle ABC equal to the angle DCE; and the angle ACB *equal* to the angle DEC; and beſides the angle BAC equal to the angle CDE: I ſay that the ſides about the equal angles of the triangles ABC, DCE are proportionals; and *that* thoſe ſides are of like ratio *which are* extended under the equal angles.

For let BC be placed in a ſtraight line with CE: and becauſe the angles ABC, ACB are leſs than two right angles; and ACB is equal to the *angle* DEC; therefore the *angles* ABC, DEC are leſs than two right angles; therefore (by com. not. 11.) BA, ED produced will meet; let them be produced and meet in F.



And becauſe the angle DCE is equal to the *angle* ABC; therefore (by 28. 1.) BF is parallel to CD: Again, becauſe the *angle* ACB is equal to DEC, AC is parallel to FE; therefore FACD is a parallelogram; therefore (by 34. 1.) FA is equal to CD; and AC to FD: and becauſe AC hath been drawn parallel to FE one of the ſides of the triangle FBE; therefore it is, (by 2. 6.) as BA *is* to AF ſo *is* BC to CE: but AF *is* equal to CD; therefore as BA is CD (by 7. 5.) ſo *is* BC to CE; and alternately (by 16. 5.) as AB *is* to BC ſo *is* DC to CE: Again, becauſe CD is parallel to BF; therefore (by 2. 6.) it is, as BC *is* to CE ſo *is* FD to DE: but FD is equal to AC: therefore as BC *is* to CE ſo is AC to DE; therefore alternately, as BC is to CA ſo *is* CE to ED: wherefore becauſe it has been demonſtrated *that* as AB *is* to BC ſo is DC to CE; and as BC *is* to CA ſo *is* CE to ED; therefore by equality (by 22. 5.) *it is*, as BA *is* to AC ſo *is* CD to DE.

Wherefore the ſides about the equal angles of equiangular triangles are proportionals; and the ſides of like ratio are extended under the equal angles. Which was to be demonſtrated.

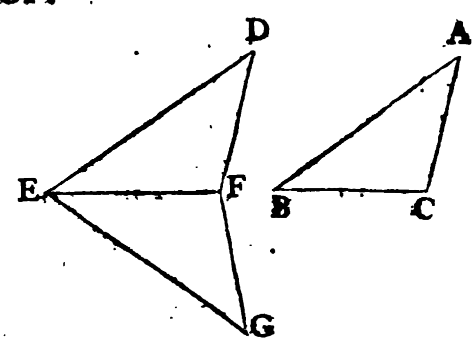
PROP.

PROP. V.

If two triangles have their ſides proportionals, the triangles will be equiangular; and will have thoſe angles equal under which the ſides of like ratio are extended.

Let ABC, DEF be two triangles having their ſides proportionals, viz. as AB *is* to BC ſo *let* DE *be* to EF; and as BC *is* to CA ſo *let* EF *be* to FD; and beſides as BA is to AC ſo *let* ED *be* to DF: I ſay, that the triangle ABC is equiangular to the triangle DEF, and will have thoſe angles equal, under which the ſides of like ratio are extended viz. ABC equal to DEF; and BCA equal to EFD; and beſides BAC equal to EDF.

For with the ſtraight line EF and at the points E, F in it, let the angle FEG be made equal to ABC; and *let* EFG be made equal to BCA; therefore (by 32. 1.) the remaining angle BAC is equal to the remaining angle EGF.



Wherefore the triangle ABC is equiangular to the triangle EGF; therefore the ſides about the equal angles of the triangles ABC, EGF are proportionals (by 4. 6.); and the ſides of like ratio are extended under the equal angles; therefore it is, as AB is to BC ſo is GE to EF: but as AB *is* to BC ſo is DE ſuppoſed *to be* to EF; wherefore (by 11. 5.) as DE is to EF ſo is GE to EF: therefore each of the *lines* DE, GE has the ſame ratio to EF; therefore (by 9. 5.) DE is equal to GE. Certainly for the ſame reaſon alſo DF is equal to GF: wherefore becauſe DE is equal to EG and EF common, certainly the two DE, EF are equal to the two GE, EF; and the baſe DF is equal to the baſe GF; therefore (by 8. 1.) the angle DEF is equal to the angle GEF; and the triangle DEF is equal to the triangle GEF; and the remaining angles *are* equal to the remaining angles, under which the equal ſides are extended (by 4. 1.): therefore the angle DFE is equal to the angle GFE; and EDF to EGF:

Book VI. EGF : and becauſe DEF is equal to GEF : but GEF (by conſt.) is equal to ABC ; therefore alſo the angle ABC is equal to DEF : Certainly for the ſame reaſon alſo ACB is equal to DFE ; and beſides (by 32. 1.) the angle at A is equal to the angle at D : Therefore the triangle ABC is equiangular to the triangle DEF.

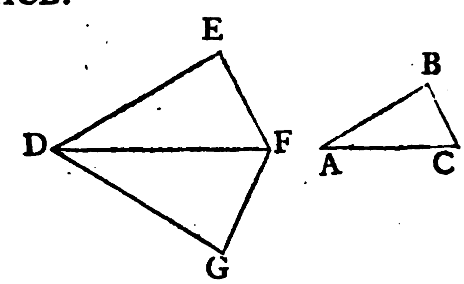
Wherefore if two triangles have their ſides proportionals, the triangles will be equiangular : and will have the angles equal, under which the ſides of like ratio are extended. Which was to be demonſtrated.

PROP. VI.

If two triangles have one angle equal to one angle, and the ſides about the equal angles proportionals : the triangles will be equiangular, and will have the angles equal, under which the ſides of like ratio are extended.

Let ABC, DEF be two triangles, having one angle BAC equal to one angle EDF ; and the ſides about the equal angles proportionals viz. as BA is to AC ſo *let* ED *be* to DF : I ſay that the triangle ABC is equiangular to the triangle DEF ; and will have the angle ABC equal to the *angle* DEF ; and ACB *equal* to DFE.

For with the ſtraight line DF and at the points, D, F in it ; let the angle FDG be made equal to either of the angles BAC, EDF ; and *let* DFG *be made equal* to ACB.



Therefore the remaining *angle* at B is equal (by 32. 1.) to the remaining *angle* at G : therefore the triangle ABC is equiangular to the triangle DGF ; therefore there is this proportion (by 4. 6.) as BA *is* to AC ſo *is* GD to DF ; but it is alſo ſuppoſed *that* as BA *is* to AC ſo *is* ED to DF ; therefore alſo (by 11. 5.) as ED *is* to DF ſo *is* GD to DF ; therefore (by 9. 5.) ED is equal to DG ; and DF is common : certainly the two ED, DF are equal to the two GD, DF ; and (by conſt.) the angle EDF is equal to the angle GDF : therefore (by 4. 1.) the baſe EF is equal to the baſe FG ; and the triangle DEF is equal to the triangle

angle DGF; and the remaining angles will be equal to the remaining angles, each to each, under which the equal ſides are extended; therefore the angle DFG is equal to the angle DFE; and the *angle* at G to the *angle* at E; but the angle DFG is equal (by conſt.) to the angle ACB; therefore alſo ACB is equal to DFE: but BAC is alſo ſuppoſed equal to EDF; therefore the remaining *angle* at B is equal to the remaining *angle* at E: wherefore the triangle ABC is equiangular to the triangle DEF. Book VI.

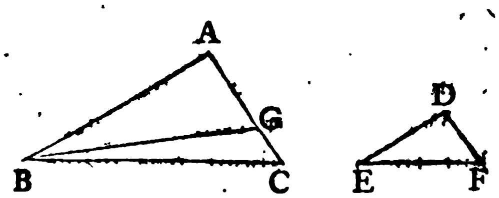
Wherefore if two triangles have one angle equal to one angle, and the ſides about the equal angles proportionals: the triangles will be equiangular, and will have the angles equal, under which the ſides of like ratio are extended. Which was to be demonſtrated.

PROP. VII.

If two triangles have one angle equal to one angle; and the ſides about *two* other angles proportionals; and if each of the remaining angles *be* at the ſame time leſs or not leſs than a right angle: the triangles will be equiangular; and will have thoſe angles equal, about which the ſides are, *which are* proportionals.

Let ABC, DEF be two triangles having one angle equal to one angle viz. the *angle* BAC *equal* to the *angle* EDF; and the ſides about *two* other angles proportionals, the *angles* ABC, DEF; ſo *that* AB *may be* to BC as DE to EF; and firſt let each of the remaining angles at C and F be at the ſame time leſs than a right angle: I ſay, that the triangle ABC is equiangular to the triangle DEF; and *that* the angle ABC will be equal to the *angle* DEF; and the remaining *angle* viz. the *angle* at C equal to the remaining *angle* at F.

For if the angle ABC be unequal to the angle DEF, one of them is greater: let ABC be the greater: and with the ſtraight line AB, and at the point B in it, let the *angle* ABG be made equal to the angle DEF.



And becauſe the angle A is equal to D; and the angle ABG to DEF; therefore the remaining angle AGB is equal to the remaining

Book VI. angle DFE: therefore the triangle ABG is equiangular to the triangle DEF; therefore it is (by 4. 6.) as AB *is* to BG so *is* DE to EF; but as DE is to EF so is AB supposed *to be* to BC; and therefore (by 11. 5.) as AB *is* to BC so *is* AB to BG: wherefore AB has the same ratio to each of the *lines* BC, BG; therefore (by 9. 5.) BC is equal to BG; so that also (by 5. 1.) the angle BGC is equal to the angle BCG: but the angle at C is supposed *to be* less than a right angle; therefore BGC is also less than a right angle; so that the angle adjacent to it viz. AGB is (by 13. 1.) greater than a right angle; and it has been demonstrated to be equal to the angle at F; therefore the angle at F is greater than a right angle; but it is supposed to be less than a right angle; which is absurd: wherefore the angle ABC is not unequal to the *angle* DEF, therefore *it is* equal: and also (by supp.) the angle at A is equal to the angle at D; therefore (by 32. 1.) the remaining *angle* at C is equal to the remaining angle at F: Therefore the triangle ABC is equiangular to the triangle DEF.

But again, let each of the angles at C and F be supposed not *to be* less than a right angle: I say again that also thus the triangle ABC is equiangular to the triangle DEF.

For the same things being constructed, we shall demonstrate in like manner that BC is equal to BG; so that also (by 5. 1.) the angle at C is equal to the *angle* BGC: but the angle at C is not less than a right angle; therefore neither is the angle BGC less than a right angle; therefore the two angles of the triangle BGC are not less than two right angles, which (by 17. 1.) is impossible: Therefore again the angle ABC is not unequal to the *angle* DEF; therefore *it is* equal: but also (by supp.) the angle at A is equal to the angle at D; therefore (by 32. 1.) the remaining *angle* at C is equal to the remaining *angle* at F: wherefore the triangle ABC is equiangular to the triangle DEF.

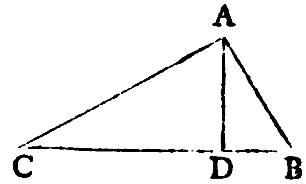
Wherefore if two triangles have one angle equal to one angle; but the sides about *two* other angles proportionals; and each of the remaining angles, at the same time, either less or not less than a right angle: the triangles will be equiangular and will have the angles equal about which the sides are, *which are* proportionals. Which was to be demonstrated.

PROP.

PROP. VIII.

If in a right angled triangle, a perpendicular be drawn from the right angle to the base; the triangles at the perpendicular are similar to the whole *triangle* and to one another.

Let ABC be a right angled triangle, having the angle BAC a right angle; and let AD be drawn from the *point* A perpendicular to BC: I say that each of the triangles ABD, ADC is similar to the whole ABC, and also to one another.



For since the angle BAC is equal to the *angle* ADB, for each of them *is* a right angle: and the angle at B is common to the two triangles, both ABC and ABD; therefore (by 32. 1.) the remaining *angle* ACB is equal to the remaining *angle* BAD; therefore the triangle ABC is equiangular to the triangle ABD; wherefore it is, (by 4. 6.) as BC subtending the right angle of the triangle ABC *is* to BA subtending the right angle of the triangle ABD, so *is* the same AB subtending the angle at C of the triangle ABC to BD subtending the angle equal to that at C, viz. the *angle* BAD of the triangle ABD: and moreover so *is* AC to AD subtending the angle at B common to the two triangles: wherefore the triangle ABC is equiangular to the triangle ABD; and has the sides about the equal angles proportionals: therefore (by 1. def. 6.) the triangle ABC is similar to the triangle ABD. Certainly in the same manner we shall demonstrate, that the triangle ADC is also similar to the triangle ABC; therefore each of the triangles ABD, ADC is similar to the whole triangle ABC.

I say also that the triangles ABD, ADC are similar to one another.

For because the right angle BDA is equal to the right angle ADC: but the *angle* BAD has been demonstrated *to be* equal to the *angle* at C; therefore also the remaining angle at B is equal to the remaining angle DAC: wherefore the triangle ABD is equiangular to the triangle ADC: therefore it is, as BD subtending the angle BAD of the triangle ABD is to DA subtending the angle at C, of the triangle ADC, equal to the angle BAD so is the same AD subtending the angle at B of the triangle ABD, to DC sub-

Book VI. tending the angle DAC of the triangle ADC, equal to the *angle* at B: and besides so is BA subtending the right angle ADB to AC subtending the right angle ADC: therefore (by 1. def. 6.) the triangle ABD is similar to the triangle ADC.

Wherefore if in a right angled triangle, a perpendicular be drawn from the right angle to the base; the triangles at the perpendicular are similar to the whole *triangle* and to one another. Which was to be demonstrated.

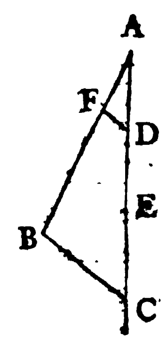
Cor. Certainly it is manifest from this, that if a perpendicular be drawn from the right angle in a right angled triangle to the base; the *perpendicular* drawn is a mean proportional between the segments of the base: and besides the side adjacent to the segment is a mean proportional between the base andanyo ne of the segments.

PROP. IX.

To cut off any part required from a given straight line.

Let AB be the given straight line: it is required to cut off any part from AB.

Let a third *part* be required; and let any straight line AC be drawn from the *point* A, containing any angle, which may accidentally happen, with the line AB; and let D be taken in AC any point which may accidentally happen; and (by 3. 1.) let DE, EC be made equal to AD; and let BC be joined; and through the point D, let DF be drawn (by 31. 1.) parallel to BC.



Wherefore because DF has been drawn *parallel* to BC one of the sides of the triangle ABC; therefore (by 2. 6.) there is this proportion, as CD *is* to DA so *is* BF to FA: but CD is the double of DA; therefore BF *is* the double of FA; therefore BA *is* the triple of AF.

Therefore the required third part AF hath been cut off from the given straight line AB. Which was to be done,

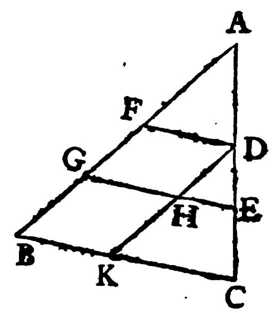
PROP.

PROP. X.

To cut an undivided ſtraight line, in like manner as a given ſtraight line hath been cut.

Let AB be the given undivided ſtraight line, and AC the one which hath been cut: it is required to cut the undivided ſtraight line AB in like manner as AC has been cut.

Let AC be cut in the points D, E: and let *the lines* be placed ſo as to contain any angle which may accidentally happen; and let BC be joined; and let DF, GE be drawn (by 31. 1.) through the *points* D, E parallel to BC: and through the *point* D, let DHK be drawn parallel to AB.



Therefore each of the *figures* FH, HB is a parallelogram: therefore (by 34. 1.) DH *is* equal to FG; and HK to GB: And becauſe HE hath been drawn *parallel* to KC one of the ſides of the triangle DKC; (by 2. 6.) there is *this* proportion, as CE *is* to ED ſo *is* KH to HD; but KH is equal to BG, and HD to GF; therefore it is (by 7. 5.) as CE is to ED ſo *is* BG to GF: Again, becauſe FD hath been drawn parallel to EG one of the ſides of the triangle AGE; therefore (by 2. 6.) there is *this proportion*, as ED *is* to DA ſo *is* GF to FA; but it has been alſo demonſtrated *that* as CE *is* to ED ſo is BG to GF: It is therefore as CE *is* to ED ſo *is* BG to GF; and as ED *is* to DA ſo *is* GF to FA.

Wherefore the given undivided ſtraight line AB hath been cut in like manner as the given ſtraight line AC had been cut. Which was to be done.

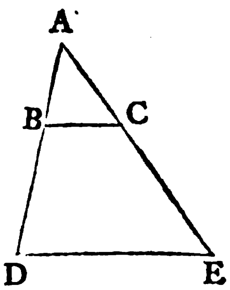
PROP. XI.

To find a third proportional to two given ſtraight lines.

Let AB, AC be the two given ſtraight lines; and let them be placed containing any angle which may accidentally happen: it is required to fiind a third proportional to AB, AC.

For let AB, AC be produced to the points D, E; and let BD be made equal to AC; and let BC be joined; and through the point D,

Book VI. D, let DE be drawn parallel to it.



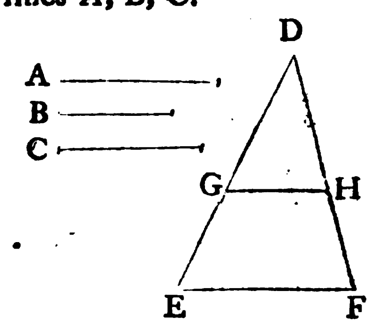
Wherefore becauſe BC hath been drawn *parallel* to DE one of the ſides of the triangle ADE, there is *this* proportion (by 2. 6.), as AB is to BD ſo *is* AC to CE: but BD is equal to AC (by conſt.); therefore it is, as AB *is* to AC ſo is AC to CE.

Wherefore two ſtraight lines AB, AC being given, CE a third proportional to them hath been found. Which was to be done.

PROP. XII.

To find a fourth proportional to three given ſtraight lines.

Let A, B, C be the three given ſtraight lines; it is required to find a fourth proportional to the ſtraight lines A, B, C.



Let two ſtraight lines DE, DF be drawn containing any angle EDF which may accidentally happen: and let (by 3. 1.) DG be made equal to A, GE equal to B; and beſides DH equal to C: and GH being joined, let EF be drawn through E parallel to it.

Therefore becauſe GH hath been drawn parallel to EF one of the ſides of the triangle DEF; therefore it is, (by 2. 6.) as DG *is* to GE ſo *is* DH to HF; but (by the conſt.) DG is equal to A, GE to B, and and DH to C; therefore it is, as A *is* to B ſo *is* C to HF.

Wherefore three ſtraight lines A, B, C being given, HF a fourth proportional has been found. Which was to be done.

PROP. XIII.

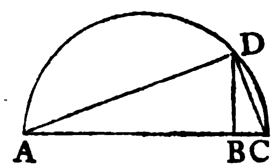
To find a mean proportional between two given ſtraight lines.

Let AB, BC be the two given ſtraight lines: it is required to find a mean proportional between AB, BC.

Let them be placed in a ſtraight line; and let the ſemicircle ADC be deſcribed upon AC; and let BD be drawn from the point B at

B at right angles to the ftraight line AC; and let AD, DC be joined. Book VI.

And becaufe the angle ADC is in a femicircle (by 31. 3.) it is a right angle: and becaufe in a right angled triangle ADC, DB hath been drawn from the right angle perpendicular to the bafe therefore (by cor. to 8. 6.) DB is a mean proportional between AB, BC the fegments of the bafe.



Wherefore two ftraight lines AB, BC being given, DB a mean proportional *between them* hath been found. Which was to be done.

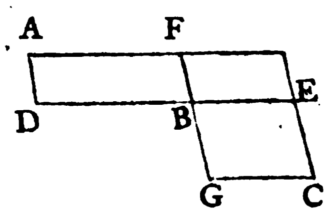
PROP. XIV.

The fides about the equal angles of equal parallelograms which have one angle equal to one angle are reciprocally proportional: and thofe parallelograms, having one angle equal to one angle, of which the fides about the equal angles are reciprocally proportional, are equal,

Let AB, BC be equal parallelograms, having the angles at B equal; and let DB, BE be placed in a ftraight line; therefore (by 14. 1.) FB, BG are in a ftraight line: I fay that the fides of the *parallelograms* AB, BC, which are about the equal angles, are reciprocally proportional, that is, as DB is to BE fo is GB to BF.

For let the parallelogram FE be compleated.

And becaufe the parallelogram AB is equal (by fupp.) to the parallelogram BC; and FE is fome other *parallelogram*; therefore (by 7. 5.) it is, as the *parallelogram* AB is to the *parallelogram* FE fo *is* the *parallelogram* BC to the *parallelogram* FE: but as the *parallelogram* AB is to the *parallelogram* FE fo (by 1. 6.) is DB to BE: and as the *parallelogram* BC is to the *parallelogram* FE fo *is* GB to BF; wherefore alfo (by 11. 5.) as DB *is* to BE fo is GB to BF; wherefore the fides of the parallelograms AB, BC, *which are* about the equal angles, are reciprocally proportional.



But let the fides about the equal angles be reciprocally proporportional, and let it be as DB *is* to BE fo *is* GB to BF: I fay that the parallelogram AB is equal to the parallelogram BC.

Book VI. For becaufe it is, as DB *is* to BE fo *is* GB to BF; but as DB *is* to BE fo (by 1. 6.) *is* the parallelogram AB to the parallelogram FE; and as GB *is* to BF fo *is* the parallelogram BC to the *parallelogram* FE; and therefore (by 11. 5.) as the *parallelogram* AB is to the *parallelogram* FE fo is the *parallelogram* BC to the *parallelogram* FE: therefore (by 9. 5.) the parallelogram AB is equal to the parallelogram BC.

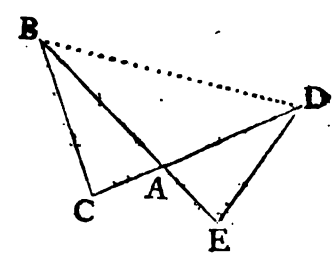
Wherefore the fides about the equal angles of equal parallelograms which have one angle equal to one angle, are reciprocally proportional: and thofe parallelograms, having one angle equal to one angle, of which the fides about the equal angles are reciprocally proportional, are equal. Which was to be demonftrated.

PROP. XV.

The fides, about the equal angles of equal triangles, which have one angle equal to one angle, are reciprocally proportional: and thofe *triangles* having one angle equal to one angle, of which the fides about the equal angles are reciprocally proportional, are equal.

Let ABC, ADE be equal triangles, having one angle equal to one *angle*; viz. *BAC* equal to DAE: I fay that the fides about the equal angles of the triangles ABC, ADE are reciprocally proportional; that is, as CA is to AD fo *is* EA to AB.

For let the *triangles* be placed in fuch a manner, that CA, AD may be in a ftraight line: therefore alfo (by 14. 1.) EA, AB are in a ftraight line: and let BD be joined.



Then becaufe (by fupp.) the triangle ABC is equal to the triangle ADE, and ABD is another *triangle*; therefore it is, as the triangle CAB is to the triangle BAD fo is the triangle ADE to the triangle BAD: but (by 1. 6.) as the *triangle* CAB *is* to BAD fo *is* CA to AD; and as the triangle EAD is to the *triangle* BAD fo *is* EA to AB; therefore alfo (by 11. 5.) as CA *is* to AD fo *is* EA to AB: therefore the fides about

about the equal angles of the triangles ABC, ADE are reciprocally proportional. Book VI.

But let the ſides of the triangles ABC, ADE be reciprocally proportional; and let it be as CA *is* to AD ſo *is* EA to AB: I ſay that the triangle ABC is equal to the triangle ADE.

For again BD being joined: becauſe it is (by ſupp.) as CA is to AD ſo *is* EA to AB; but as CA is to AD (by 1. 6.) ſo *is* the triangle ABC to the triangle BAD; and as EA *is* to AB ſo is the triangle EAD to the triangle BAD: wherefore (by 11. 5.) as the triangle ABC is to the *triangle* BAD ſo is the triangle EAD to the *triangle* BAD: therefore each of the *triangles* ABC, ADE have the ſame ratio to BAD; therefore (by 9. 5.) the triangle ABC is equal to the triangle EAD.

Wherefore the ſides about the equal angles of equal triangles, which have one angle equal to one angle, are reciprocally proportional: and thoſe *triangles* having one angle equal to one angle, of which the ſides about the equal angles are reciprocally proportional, are equal. Which was to be demonſtrated.

PROP. XVI.

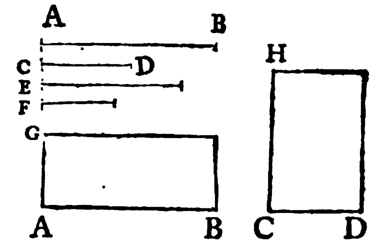
If four ſtraight lines be proportionals, the rectangle contained by the extremes is equal to the rectangle contained by the means: and if the rectangle contained by the extremes be equal to the rectangle contained by the means, the four ſtraight lines will be proportionals.

Let the four ſtraight lines AB, CD, E, F be proportionals, viz. as AB *is* to CD ſo *let* E *be* to F: I ſay that the rectangle contained by AB *and* F is equal to the rectangle contained by CD *and* E.

For from the points A, C let AG, CH be drawn at right angles to the ſtraight lines AB, CD; and let AG be made equal to F; and CH equal to E; and let the parallelograms BG, DH be compleated.

And becauſe it is, as AB *is* to CD ſo *is* E to F; and CH *is* equal to E and AG to F: therefore it is (by 7. 5.) as AB *is* to CD ſo *is* CH to AG; therefore the ſides about the equal angles of the parallelograms BG, DH are reciprocally porportional; but thoſe equi-

Book VI. angular parallelograms are equal (by 14. 6.) of which the sides about the equal angles are reciprocally proportional: therefore the parallelogram BG is equal to the parallelogram DH: and the *parallelogram* BG is the *rectangle contained* by AB and F; for AG is equal to F: and the *parallelogram* DH is the rectangle contained by CD *and* E; for CH is equal to E: wherefore the rectangle contained by AB *and* F is equal to the rectangle contained by CD *and* E.



But let the rectangle contained by AB *and* F be equal to the rectangle contained by CD *and* E: I say that the four straight lines will be proportionals, viz. as AB *is* to CD so *will* E *be* to F. For the same things being constructed; because (by the supp.) the *rectangle contained* by AB, F is equal to the *rectangle contained* by CD, E; and the *parallelogram* BG is the *rectangle contained* by AB, F; for AG is equal to F: and the *parallelogram* DH is the rectangle contained by CD, E; for CH is equal to E: therefore the parallelogram BG is equal to the parallelogram DH; and they are equiangular: but (by 14. 6.) the sides about the equal angles of equal and equiangular parallelograms are reciprocally proportional: wherefore it is, as AB is to CD so is CH to AG; but CH is equal to E; and AG to F: therefore it is as AB *is* to CD so *is* E to F.

Wherefore if four straight lines be proportionals, the rectangle contained by the extremes is equal to the rectangle contained by the means; and if the rectangle contained by the extremes be equal to the rectangle contained by the means, the four straight lines will be proportionals. Which was to be demonstrated.

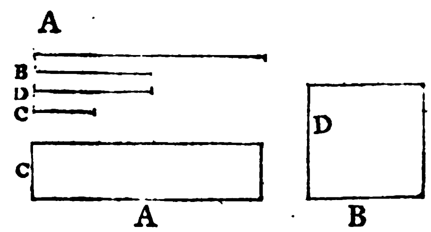
PROP. XVII.

If three straight lines be proportionals; the rectangle contained by the extremes is equal to the square of the mean *proportional*: and if the rectangle contained by the extremes be equal to the square of the mean *term*, the three straight lines will be proportionals.

Let

Let the three straight lines A, B, C be proportionals; viz. as A is to B so *let* B *be* to C: I say that the rectangle contained by A *and* C is equal to the square of the mean B. Book VI.

Let D be made equal to B.



And because it is, as A *is* to B so *is* B to C; and B is equal to D; therefore it is, as A is to B so *is* D to C: but if four straight lines be proportionals, the rectangle contained by the extremes is equal (by 16. 6.) to the rectangle contained by the means; therefore the rectangle contained by A *and* C is equal to the rectangle contained by B *and* D: but the rectangle contained by B *and* D is equal to the *square* of B; for B is equal to D: therefore the rectangle contained by A *and* C is equal to the *square* of B.

But let the *rectangle contained* by A and C be equal to the *square* of B; I say that it is, as A *is* to B so *is* B to C.

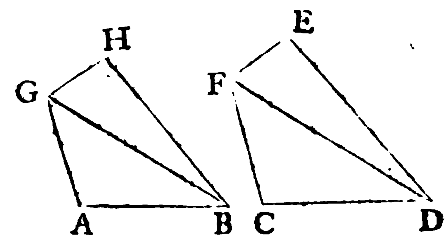
For the same things being constructed; because (by supp.) the rectangle contained by A *and* C is equal to the *square* of B; but the *square* of B is equal to the *rectangle contained* by B *and* D; for B is equal to D: therefore the *rectangle contained* by A *and* C is equal to the *rectangle contained* by B *and* D; but (by 16. 6.) if the *rectangle contained* by the extremes be equal to the rectangle contained by the means; the four straight lines are proportionals: therefore it is, as A *is* to B so *is* D to C; but B is equal to D; therefore as A *is* to B so *is* B to C.

Wherefore if three straight lines be proportionals; the rectangle contained by the extremes is equal to the square of the mean *proportional*: and if the rectangle contained by the extremes be equal to the square of the mean *term*; the three straight lines will be proportionals. Which was to be demonstrated.

PROP. XVIII.

Upon a given straight line to describe a rectilineal *figure*, similar and similarly situated to a given rectilineal *figure*.

Book VI. Let AB be the given ftraight line; and CE the given rectilineal *figure*: it is required, upon the given ftraight line AB to defcribe a rectilineal figure fimilar and fimilarly fituated to the rectilineal figure CE.



Let DF be joined; and with the ftraight line AB and at the points A, B in it, let the *angle* GAB be made equal to the angle at C; and the *angle* ABG *equal* to the *angle* CDF; therefore (by the 32. 1.) the remaining *angle* CFD is equal to the remaining *angle* AGB; therefore the triangle FCD is equiangular to the triangle GAB: therefore there is *this* proportion (by 4. 6.) as FD *is* to GB fo *is* FC to GA; and fo *is* CD to AB: Again, with the ftraight line BG, and at the points B, G in it, let the *angle* BGH be made equal to the angle DFE; and the *angle* GBH equal to the *angle* FDE; therefore the remaining *angle* at H is equal to the remaining *angle* at E: therefore the triangle FDE is equiangular to the triangle GBH; therefore there is this proportion (by 4. 6.) as FD *is* to GB fo *is* FE to GH; and *fo is* ED to HB: But it has been alfo demonftrated *that* as FD is to GB fo is FC to GA, and *fo is* CD to AB: therefore (by 11. 5.) as FC is to GA fo *is* CD to AB and *fo is* FE to GH, and befides *fo is* ED to HB: And becaufe the angle CFD is equal to the angle AGB and DFE *is equal* to BGH; therefore the whole *angle* CFE is equal to the whole AGH. Certainly for the fame reafon alfo the *angle* CDE is equal to ABH: but alfo the *angle* at C is equal to the *angle* at A; and the *angle* at E to the *angle* at H: therefore the *figure* AH is equiangular to the *figure* CE; and has the fides about the equal angles in it proportionals; therefore (by 1. def. 6.) the rectilineal *figure* AH is fimilar to the rectilineal *figure* CE.

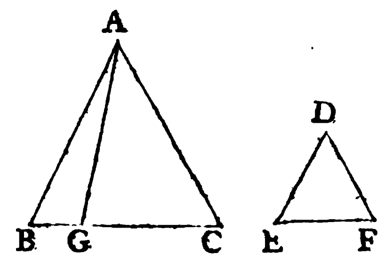
Wherefore upon a given ftraight line AB a rectilineal *figure* AH hath been defcribed, fimilar and fimilarly fituated to the given rectilineal *figure* CE. Which was to be done.

PROP. XIX.

Similar triangles are to one another in the duplicate ratio of the fides of like ratio. Let

Let ABC, DEF be similar triangles, having the angle at B equal to the *angle* at E: and as AB *is* to BC so *let* DE *be* to EF; so that (by 12. def. 5.) BC may be the side of like ratio to EF: I say that the triangle ABC has to the triangle DEF the duplicate ratio of that which BC has to EF. Book VI.

For (by 11. 6.) let BG be taken a third proportional to BC, EF; so that it may be, as BC is to EF so *is* EF to BG: and let GA be joined,



Wherefore because it is, as AB *is* to BC so *is* DE to EF; therefore alternately (by 16. 5.) it is, as AB *is* to DE so *is* BC to EF: but (by const.) as BC *is* to EF so *is* EF to BG: therefore also (by 11. 5.) as AB *is* to DE so *is* EF BG: therefore the sides about the equal angles of the triangles ABG, DEF are reciprocally proportional: but those *triangles* having one angle equal to one angle, and of which the sides about the equal angles are reciprocally proportional, are equal. Therefore the triangle ABG is equal to the triangle DEF: and because it is, as BC *is* to EF so *is* EF to BG; and if three straight lines be proportionals (by 10. def. 5.) the first is said to have to the third a duplicate ratio of that which it *has* to the second: therefore BC has to BG a duplicate ratio of that which BC *has* to EF: but as BC *is* to BG so (by 1. 6.) *is* the triangle ABC to the triangle ABG: therefore also the triangle ABC has to the triangle ABG the duplicate ratio of that which BC *has* to EF: but the triangle ABG is equal to the triangle DEF; therefore (by 7. 5.) the triangle ABC has to the triangle DEF the duplicate ratio of that which BC *has* to EF.

Wherefore similar triangles are to one another in the duplicate ratio of the sides of like ratio. Which was to be demonstrated.

Cor. Certainly from this it is manifest, that if three straight lines be proportionals, it is, as the first *is* to the third so *is* a triangle described upon the first, to a similar and similarly situated *triangle described* upon the second; since it has been demonstrated, as CB *is* to BG so *is* the triangle ABC to the triangle ABG, that is DEF.

PROP.

Book VI.

PROP. XX.

Similar polygons are divided into similar triangles, and into the same number, and *have* the same ratio *to one another which* the whole *polygons have*: and the *one* polygon has to the *other* polygon the duplicate ratio of that which one side of the one has to the side of like ratio of the other.

Let ABCDE, FGHKL be similar polygons, and let AB be the side of like ratio FG: I say that the polygons ABCDE, FGHKL are divided into similar triangles; and into the same number; and have the same ratio *to one another which* the whole *polygons have*; and *that* the polygon ABCDE has to the polygon FGHKL the duplicate ratio of that which AB *has* to FG.

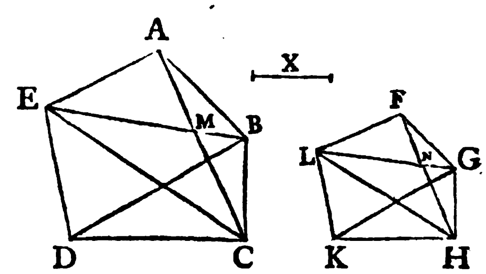
Let BE, EC; GL, LH be joined.

And because the polygon ABCDE is similar to the polygon FGHKL, the angle BAE is equal to the *angle* GFL: and it is, as BA *is* to AE so *is* GF to FL: wherefore because there are two triangles ABE, FGL having one angle equal to one angle, and the sides about the equal angles proportionals; therefore (by 6. 6.) the triangle ABE is equiangular to the triangle FGL; so that (by 4. 6.) it is also similar: therefore the angle ABE is equal to the angle FGL: and also on account of the similarity of the polygons, the whole *angle* ABC is equal to the whole *angle* FGH; therefore the remaining angle EBC is equal to the remaining *angle* LGH; and because, on account of the similarity of the triangles ABE, FGL, it is, as EB *is* to BA so *is* LG to GF; but also, on account of the similarity of the polygons, it is, as AB *is* to BC so *is* FG to GH: therefore by equality it is, (by 22, 5.) as EB *is* to BC so *is* LG to GH; and the sides *therefore* about the equal angles EBC, LGH are proportionals; therefore the triangle EBC is equiangular to the triangle LGH (by 6. 6.); so that *it is* also similar (by 4. 6.): Certainly for the same reason the triangle ECD is similar to the triangle LHK: therefore the similar polygons ABCDE, FGHKL are divided into similar triangles, and into an equal number.

I say also that they have the same ratio to one another which the whole polygons have: that is, so that the triangles are proportionals; and that the antecedents are the *triangles* ABE, EBC, ECD

ECD and the consequents of them, the *triangles* FGL, LGH, LHK: also that the polygon ABCDE has to the polygon FGHKL the duplicate ratio of that which the side of like ratio *of the one* has to the side of like ratio *of the other*, that is, which AB *has* to FG. Book VI.

For let AC, FH be joined.



And because, on account of the similarity of the polygons, the angle ABC is equal to the *angle* FGH, and it is as AB *is* to BC so is FG to GH; the triangle ABC is equiangular (by 6. 6.) to the triangle FGH: therefore the *angle* BAC is equal to the *angle* GFH; and BCA to GHF: and because the angle BAM is equal to GFN; and the angle ABM has been demonstrated *to be* equal to FGN; therefore the remaining *angle* AMB is equal (by 32. 1.) to the remaining *angle* FNG; therefore the triangle ABM is equiangular to the triangle FGN: Certainly in the same manner we shall demonstrate that the *triangle* BMC is equiangular to the triangle GNH; therefore there is this proportion, (by 4. 6.) as AM *is* to MB so *is* FN to NG; and as BM *is* to MC so *is* GN to NH: so that also by equality (by 22. 5.) as AM is to MC so *is* FN to NH: but as AM *is* to MC so (by 1. 6.) *is* the triangle ABM to the *triangle* BMC; and *so is the triangle* AEM to the *triangle* MEC; for they are to one another as *their* bases: And (by 12. 5.) as one of the antecedents is to one of the consequents so *are* all the antecedents to all the consequents; therefore as the triangle AMB is to the triangle BMC so *is* the *triangle* ABE to the *triangle* CBE: but as the *triangle* AMB *is* to the *triangle* BMC so is AM to MC: therefore also (by 11. 5.) as AM *is* to MC so *is* the triangle ABE to the triangle CBE. Certainly for the same reason also as FN is to NH so is the triangle FGL to the triangle GLH: and it is, as AM *is* to MC so *is* FN to NH; therefore (by 11. 5.) as the triangle ABE *is* to the triangle BEC so *is* the triangle FGL to the triangle GHL; and alternately (by 16. 5.) as the triangle ABE *is* to the triangle FGL so *is* the triangle BEC to the triangle GLH.

Cer-

Book VI. Certainly in the ſame manner we ſhall demonſtrate, BD and GK being joined, that alſo, as the triangle BEC is to the triangle GLH ſo is the triangle ECD to the triangle LHK: and becauſe it is, as the triangle ABE *is* to the triangle FGL ſo is the triangle EBC to the triangle LGH, and beſides ſo *is* the triangle ECD to the triangle LHK: wherefore alſo (by 12. 5.) as one of the antecedents is to one of the conſequents, ſo *are* all the antecedents to all the conſequents; therefore as the triangle ABE is to the triangle FGL ſo is the polygon ABCDE to the polygon FGHKL: but the triangle ABE has to the triangle FGL (by 19. 6.) the duplicate ratio of that which the ſide of like ratio AB has to the ſide of like ratio FG; for ſimilar triangles are in the duplicate ratio of the ſides of like ratio; therefore alſo the polygon ABCDE has to the polygon FGHKL the duplicate ratio of that which the ſide of like ratio AB *has* to the ſide of like ratio FG.

Wherefore ſimilar polygons are divided into ſimilar triangles, and into an equal number of them, and of like ratio with the whole polygons; and the *one* polygon has to the *other* polygon the duplicate ratio of that which *one* ſide of like ratio has to another ſide of like ratio. Which was to be demonſtrated.

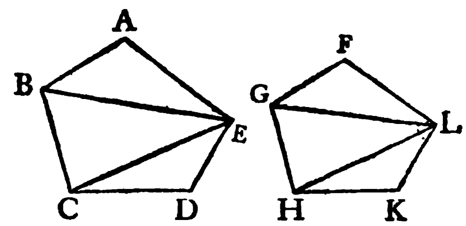
Cor. 1. Certainly in like manner it will be demonſtrated in quadrilateral figures, that they are in the duplicate ratio of the ſides of like ratio; and it has been demonſtrated alſo in triangles; ſo that univerſally ſimilar rectilineal figures are to one another in the duplicate ratio of the ſides of like ratio.

Cor. 2. And if we take the *line* X (by 11. 6.) a third proportional to AB, FG; (by 10. def. 5.) AB has to X the duplicate ratio of that which AB *has* to FG: and alſo a polygon to a polygon, and a quadrilateral *figure* to a quadrilateral *figure* has the duplicate ratio of that which the ſide of like ratio has to the ſide of like ratio, that is, *which* AB *has* to FG: and this has been demonſtrated alſo in triangles; ſo that it *is* alſo manifeſt univerſally, that if three ſtraight lines be proportionals, it will be, as the firſt *is* to the third ſo is a *rectilineal* figure deſcribed upon the firſt to a ſimilar and ſimilarly ſituated *one deſcribed* upon the ſecond.

OTHERWISE. We ſhall demonſtrate in a different manner and more expeditiouſly, that the triangles are of like ratio.

For

For again let the polygons ABCDE, FGHKL be ſuppoſed to be deſcribed; and let BE, EC; GL, LH be joined: I ſay that it is, as the triangle ABE *is* to the triangle FGL ſo *is* the *triangle* EBC to the *triangle* LGH, and CDE to HKL.

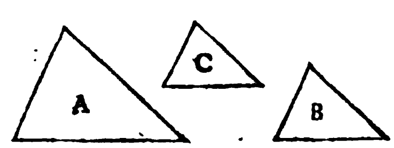


For becauſe the triangle ABE is ſimilar to the triangle FGL; therefore (by 19. 6.) the triangle ABE has to the triangle FGL the duplicate ratio of that which BE *has* to GL. Certainly for the ſame reaſon alſo the triangle BEC has to the triangle GLH the duplicate ratio of that which BE has to GL: therefore it is (by 11. 5.) as the triangle ABE is to the *triangle* FGL ſo is the *triangle* EBC to the *triangle* LGH. Again, becauſe the triangle EBC is ſimilar to the triangle LGH; therefore EBC has to LGH the duplicate ratio of that which the ſtraight line CE has to the *ſtraight line* HL. Certainly for the ſame reaſon alſo, the triangle ECD has to the triangle LHK the duplicate ratio of that which CE *has* to HL; therefore it is, (by 11. 5.) as the triangle EBC *is* to the *triangle* LGH ſo *is* the *triangle* ECD to the *triangle* LHK: but it has been demonſtrated *that* as the *triangle* EBC *is* to LGH ſo is ABE to FGL; therefore alſo as ABE *is* to FGL ſo *is* BEC to LGH, and ſo *is* ECD to LHK: wherefore alſo (by 12. 5.) as one of the antecedents is to one of the conſequents ſo are all the antecedents to all the conſequents; and the remainder as in the former demonſtration. Which was to be demonſtrated.

PROP. XXI.

Figures which are ſimilar to the ſame rectilineal figure; are alſo ſimilar to one another.

For let each of the rectilineal figures A, B be ſimilar to C: I ſay that A is alſo ſimilar to B.



For becauſe A is ſimilar to C, it is alſo equiangular to it, and has the ſides about the equal angles proportionals (by 1. def. 6.). Again be-

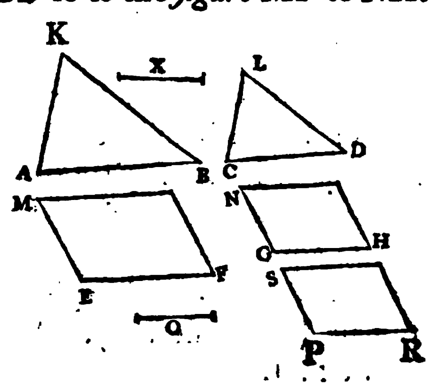
Book VI. cause B is similar to C; it is also equiangular to it, and has the sides about the equal angles proportionals: therefore each of the *figures* A, B are equiangular to C and have the sides about *their* equal angles proportionals *to the sides about the equal angles of the figure C*; wherefore the *figure* A is equiangular to B (by com. not. 1.) and has (by 11. 5.) the sides about the equal angles proportionals: wherefore (by 1. def. 6.) *the figure* A is similar to the *figure* B. Which was to be demonstrated.

PROP. XXII.

If four straight lines be proportionals; also the rectilineal figures similar and similarly described upon them will be proportionals: and if the rectilineal figures similar and similarly described upon them be proportionals; also the straight lines themselves will be proportionals.

Let the four straight lines AB, CD, EF, GH be proportionals, viz. as AB *is* to CD so *let* EF *be* to GH; and let the rectilineal figures KAB, LCD be described upon AB, CD similar and similarly situated; and the rectilineal figures MF, NH upon EF, GH similar and similarly situated: I say that it is, as KAB *is* to LCD so *is* MF to NH.

For let X be taken (by 11. 6.) a third proportional to AB, CD: and O a third proportional to EF, GH: and because it is, as AB *is* to CD so *is* EF to GH; and as CD *is* to X so *is* GH to O; therefore by equality it is, (by 22. 5.) as AB is to X so *is* EF to O: But as AB is to X so (by cor. 2. to 20. 6.) is the *figure* KAB to LCD; and as EF is to O so is the *figure* MF to NH; therefore (by 11. 5.) as the *figure* KAB is to LCD so is the *figure* MF to NH.



But let it be, as the figure KAB is to LCD so *let* the figure MF *be* to NH: I say that it is, as AB *is* to CD so *is* EF to GH.

For as AB is to CD so let EF be made (by 12. 6.) to PR; and let the rectilineal figure SR be described (by 18. 6.) upon PR similar and similarly situated to either of the figures MF, NH.

Wherefore because it is, as AB *is* to CD so *is* EF to PR; and Book VI.
the *figures* KAB, LCD have been described upon AB *and* CD similar and similarly situated; and the figures MF *and* SR similar and similarly situated upon EF *and* PR; therefore it is (by part. 1. of this.) as the *figure* KAB is to LCD so *is* the *figure* MF to SR; but it is also supposed as the *figure* KAB *is* to LCD so *is* the *figure* MF to NH; therefore (by 11. 5.) the *figure* MF has the same ratio to each of the *figures* NH and SR; therefore (by 9. 5.) NH is equal to SR; but it is similar to it and similarly situated: therefore (by the following lemma) GH is equal to PR; and because it is, as AB is to CD so is EF to PR (by const.): and PR is equal to GH; therefore it is, as AB *is* to CD so *is* (by 7. 5.) EF to GH.

Wherefore if four straight lines be proportionals; also the similar rectilineal figures similarly described upon them will be proportionals: and if the similar figures similarly described upon them be proportionals; also the straight lines themselves will be proportionals. Which was to be demonstrated.

LEMMA. If rectilineal figures be equal and similar, we shall thus demonstrate, that the sides of them, which are of like ratio, are equal.

Let NH and SR be equal and similar rectilineal figures; and let it be, as HG *is* to GN so *is* RP to PS: I say that PR is equal to HG.

For if they are unequal, one of them is greater; let RP be greater than HG: and because it is, as RP *is* to PS so *is* HG to GN, and alternately, as RP is to HG so is PS to GN: but PR is greater than GH; therefore also PS is greater than GN: so that also (by 20. 6.) the *figure* RS is greater than the *figure* HN; but it is also equal, which is impossible; therefore PR is not unequal to GH; therefore equal. Which was to be demonstrated.

PROP. XXIII.

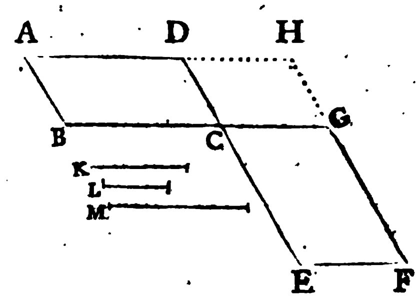
Equiangular parallelograms have to one another the ratio compounded *of the ratios* of their sides.

Let AC, CF be equiangular parallelograms, having the angle BCD equal to the *angle* ECG: I say that the parallelogram AC

 has

Book VI. has to the parallelogram CF the ratio compounded *of the ratios* of the ſides; of that which BC has to CG; and of that which DC has to CE.

For let them be placed ſo that BC may be in a ſtraight line with CG; therefore (by 14. 1.) DC is in a ſtraight line with CE: and let the parallelogram DG be compleated; and let K any ſtraight line be drawn; and, as BC *is* to CG ſo (by 12. 6.) let K be made to L; and as DC is to CE ſo *let* L *be made* to M.



Therefore (by conſt.) the ratios of K to L and of L to M, are the ſame *ratios* with the ratios of the ſides; viz. of BC to CG and of DC to CE: but the ratio of K to M is compounded both of the ratio of K to L and *of the ratio* of L, to M; ſo that alſo K has to M the ratio compounded of *the ratios* of the ſides.

And becauſe it is (by 1. 6.) as BC *is* to CG ſo *is* the parallelogram AC to the *parallelogram* CH: but as BC *is* to CG ſo (by conſt.) *is* K to L; therefore alſo (by 11. 5.) as K is to L ſo is the *parallelogram* AC to the *parallelogram* CH: Again becauſe it is, (by 1. 6.) as DC is to CE ſo is the parallelogram CH to the *parallelogram* CF: but as DC is to CE ſo (by conſt.) is L to M; therefore alſo (by 11. 5.) as L is to M ſo is the *parallelogram* CH to the *parallelogram* CF: wherefore becauſe it has been demonſtrated, *that* as K is to L ſo is the parallelogram AC to the parallelogram CH; and as L is to M ſo *is* the parallelogram CH to the *parallelogram* CF; therefore by equality it is, (by 22. 5.) as K is to M ſo is the parallelogram AC to the *parallelogram* CF; and K has to M the ratio compounded *of the ratios* of the ſides; therefore alſo, the *parallelogram* AC has to the *parallelogram* CF the ratio compounded *of the ratios* of the ſides.

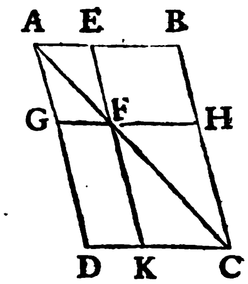
Wherefore equiangular parallelograms have to one another the ratio compounded *of the ratios* of their ſides. Which was to be demonſtrated.

PROP.

PROP. XXIV.

The parallelograms, which are about the diameter of every parallelogram, are ſimilar to the whole and to one another.

Let ABCD be a parallelogram ; and AC its diameter ; and let EG, HK be parallelograms about the *diameter* AC : I ſay that each of the parallelograms EG, HK is ſimilar to the whole *parallelogram* ABCD and to one another.



For becauſe EF has been drawn parallel to BC one of the ſides of the triangle ABC, there is this proportion (by 2. 6.), as BE *is* to EA ſo *is* CF to FA : again becauſe FG hath been drawn parallel to CD one of the ſides of the triangle ADC ; therefore there is this proportion, as CF *is* to FA ſo *is* DG to GA : but as CF is to FA ſo has BE been demonſtrated *to be* to EA : therefore alſo (by 11. 5.) as BE is to EA ſo is DG to GA ; and by compoſition therefore (by 18. 5.) as BA *is* to AE ſo *is* DA to AG ; and alternately (by 16. 5.) as BA *is* to AD ſo *is* EA to AG : therefore the ſides about the common angle BAD of the parallelograms ABCD *and* EG are proportionals. And becauſe GF is parallel to DC ; the angle AGF (by 29. 1.) is equal to ADC ; and the angle GFA to DCA ; and the angle DAC is common to the two triangles ADC, AGF ; therefore the triangle ADC is equiangular to the triangle AGF. Certainly for the ſame reaſon alſo the triangle ABC is equiangular to the triangle AFE : therefore alſo the whole parallelogram ABCD is equiangular to the parallelogram EG : therefore there is this proportion, (by 4. 6.) as AD *is* to DC ſo *is* AG to GF ; and as DC *is* to CA ſo *is* GF to FA ; and as AC *is* to CB ſo *is* AF to FE and beſides as CB *is* to BA ſo *is* FE to EA : and becauſe it has been demonſtrated *that* as DC *is* to CA ſo *is* GF to FA ; and as AC *is* to CB ſo *is* AF to FE ; therefore by equality it is, (by 22. 5.) as DC is to CB ſo is GF to FE : therefore the ſides about the equal angles of the parallelograms ABCD and EG are proportionals : therefore (by 1. def. 6.) the parallelogram ABCD is ſimilar to the parallelogram EG. Certainly for the ſame reaſon alſo the parallelogram ABCD is ſimilar to the parallelogram HK ; there-

Book VI. therefore each of the parallelograms EG, HK is ſimilar to the parallelogram ABCD: but (by 21. 6.) *figures* which are ſimilar to the ſame rectilineal *figure* are alſo ſimilar to one another: therefore alſo the parallelogram EG is ſimilar to the parallelogram HK.

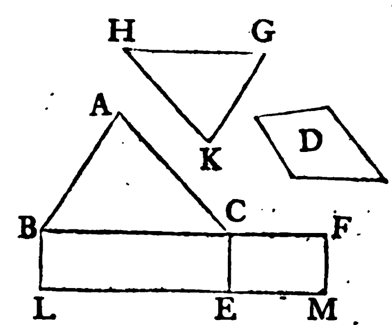
Wherefore the parallelograms which are about the diameter of every parallelogram are ſimilar to the whole, and to one another. Which was to be demonſtrated.

PROP. XXV.

To make the ſame *rectilineal figure* ſimilar to a given rectilineal figure, and equal to another given *rectilineal figure.*

Let ABC be the given rectilineal figure, to which it is required that the *figure* ſhould be made ſimilar; and D the *one* to which *it is to be* equal: it is required that the ſame figure be made ſimilar to ABC and equal to D.

For let the parallelogram BE be applied to the ſtraight line BC (by 44. 1.) equal to the triangle ABC; and (by 45. 1.) *let* the parallelogram CM *be applied* to the ſtraight line CE, equal to the *figure* D, in the angle FCE which is equal to the *angle* CBL; therefore BC is (by 14. 1.) in a ſtraight line with CF; and LE with EM; and let GH be taken (by 13. 6.) a mean proportianal between BC, CF: and let the *figure* KGH be deſcribed (by 18. 6.) upon GH ſimilar to ABC, and ſimilarly ſituated.



And becauſe it is, as BC *is* to GH ſo *is* GH to CF (by conſt.); and (by 2. Cor. to 20. 6.) if three ſtraight lines be proportionals, it is, as the firſt *is* to the third ſo *is* the *rectilineal* figure upon the firſt to a ſimilar and ſimilarly deſcribed *figure* upon the ſecond; therefore it is, as BC *is* to CF ſo *is* the triangle ABC to KGH; but alſo (by 1. 6.) as BC *is* to CF ſo *is* the parallelogram BE to the parallelogram EF; therefore alſo (by 11. 5.) as the triangle ABC *is* to the triangle KGH ſo *is* the parallelogram BE to the parallelogram EF: alternately (by 16. 5.) there-

therefore as the triangle ABC *is* to the parallelogram BE so *is* the triangle KGH to the parallelogram EF: but (by conft.) the triangle ABC *is* equal to the parallelogram BE; therefore also the triangle KGH *is* equal to the parallelogram EF: but the parallelogram EF is equal to the *rectilineal figure* D: therefore also KGH is equal to D; and KGH is also similar to ABC. Book VI.

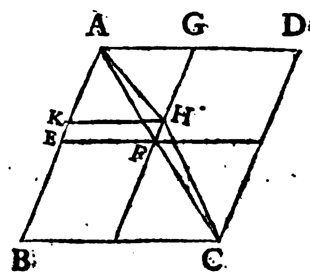
Wherefore, the same *figure* KGH has been made similar to the given rectilineal *figure* ABC and equal to another given figure. Which was to be done.

PROP. XXVI.

If a parallelogram be taken away from a parallelogram similar to the whole and similarly situated, having a common angle with it; it is about the same diameter with the whole *parallelogram*.

For let the parallelogram AEFG be taken away from the parallelogram ABCD, similar to ABCD, and similarly situated, having the angle DAB common with it: I say that ABCD is about the same diameter with AEFG.

For *if* not, but if possible, let AHC be the diameter [of them] *of the parallelogram BD*; and let HK be drawn through the point H parallel to either of the *lines* AD, BC.



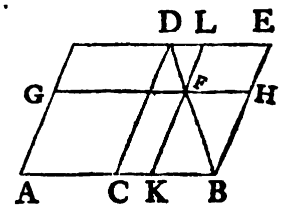
Wherefore because ABCD is about the same diameter with the *parallelogram* KG; ABCD is (by 24. 6.) similar to the *parallelogram* KG; therefore it is (by 1. def. 6.) as DA *is* to AB so *is* GA to AK; but it is also, on account of the similarity of the *parallelograms* ABCD, EG as DA *is* to AB so *is* GA to AE; therefore also (by 11. 5.) as GA is to AE so is GA to AK: therefore GA has the same ratio to each of the *lines* AE, AK; therefore (by 9. 5.) AE is equal to AK; the less to the greater; which is impossible: therefore ABCD is not about the same diameter with the *parallelogram* KG: therefore the parallelogram ABCD is about the same diameter with the parallelogram AEFG.

Wherefore if a parallelogram be taken away from a parallelogram similar to the whole and similarly situated, having a common angle with

Book VI. with it; it is about the same diameter with the whole *parallelogram*. Which was to be demonstrated.

PROP. XXVII.

Of all the parallelograms applied to the same straight line, and deficient by parallelogram figures similar and similarly situated to the *parallelogram* applied to half the line: the parallelogram applied to half the line, *as* being similar to the defect, is the greatest.



Let AB be a straight line; and let it be cut in halves in the point C: and let the parallelogram AD be applied to the straight line AB; deficient by the parallelogram figure CE, similar and similarly situated to that described upon the half of the *straight line* AB, that is upon BC: I say that AD is the greatest of all the parallelograms applied to the straight line AB, and deficient by parallelogram figures similar and similarly situated to CE. For let the parallelogram AF be applied to the straight line AB, deficient by the parallelogram figure KH, similar and similarly situated to CE; I say that AD is greater than AF.

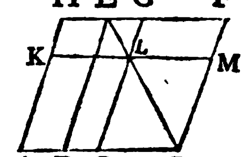
For because the parallelogram CE is similar to the parallelogram KH, they are (by 26. 6.) about the same diameter: let their diameter DB be drawn: and let the figure be described.

Wherefore because CF is equal (by 43. 1.) to FE, let KH which is common be added, therefore the whole CH is equal to the whole KE: but CH is equal to CG, because AC is equal to CB; therefore also CG is equal to EK: let CF which is common be added; therefore the whole AF is equal to the gnomon CHL: so that the parallelogram CE, that is AD (by 36. 1.) is greater than the parallelogram AF.

But again let the straight line AB be cut in halves in the point C; and AL being applied deficient by the figure CM; and again let the parallelogram AE be applied to AB deficient by DF similar and similarly situated to CM the *parallelogram* applied to half the *line* AB: I say that AL the *parallelogram* applied to half the line is greater than AE.

For

For becaufe DF is fimilar to GM, they are (by 26. 6.) about the fame diameter; let their diameter be EB; and let the figure be defcribed. Book VI.



And becaufe LF is equal to LH (by 36. 1.) becaufe FG is equal to GH; therefore LF is equal to DL; therefore DL is greater than EK; let DK *which is* common be *added*; therefore the whole AL is greater than the whole AE.

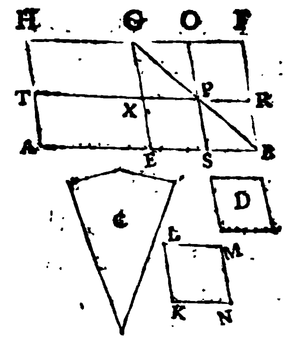
Wherefore of all the parallelograms applied to the fame ftraight line, and deficient by parallelogram figures fimilar and fimilarly fituated to the *parallelogram* applied to half the line: the parallelogram applied to half the line, *as* being fimilar to the defect, is the greateft. Which was to be demonftrated.

PROP. XXVIII.

To apply to a given ftraight line a parallelogram equal to a given rectilineal figure, and deficient by a parallelogram figure, which is fimilar to a given parallelogram: but it is required, that the given rectilineal figure to which it is required that the one to be applied is to be equal, fhould not be greater than the parallelogram applied to half the line; thefe being fimilar, the deficient *parallelogram*, the one applied to half the line and the parallelogram to which the deficient one is required to be fimilar.

Let AB be the given ftraight line; and C the given rectilineal figure, to which it is required to apply to AB one equal; not being greater than the parallelogram applied to half the line, the defects being fimilar; and let D be the *parallelogram* to which the defect is required to be fimilar: it is required to apply to the given ftraight line AB a parallelogram equal to the given rectilineal figure C; deficient by a parallelogram figure which is fimilar to D.

Let AB be cut in halves at the point E; and let EBFG be defcribed upon EB fimilar to D, and fimilarly fituated: and let the parallelogram AG be compleated: certainly AG is either equal to C, or greater than it, on account of the limitation: if therefore AG be equal to C, what was required hath been done; for the parallelogram AG hath been applied to the given ftraight line AB, equal to the given rectilineal figure C, deficient by the paral-



Book VI. lelogram figure EF, which is similar to D: But if not; HE is greater than C: but HE is equal to EF; therefore also EF is greater than C: let the *parallelogram* KLMN be made (by 25. 6.) equal to that excess by which EF is greater than C, and similar to D, and the same similarly situated: but D is similar (by const.) to EF; therefore KM is also similar to EF; therefore let LK be the side of like ratio with GE; and LM the *side of like ratio* with GF; and because the *parallelogram* EF is equal to *the figure* C and KM together; therefore EF is greater than KM; therefore (by 1. cor. to 20. 6.) GE is greater than KL and GF than LM: Let GX be made equal to KL and GO equal to LM; and let the parallelogram XGOP be compleated; therefore XO is equal and similar to KM: but KM is similar to EF; therefore (by 21. 6.) XO is also similar to EF; therefore XO and EF (by 26. 6.) are about the same diameter: let GPB be their diameter, and let the figure be compleated.

Wherefore because the parallelogram EF is equal to C and KM together; the *parts* of which XO and KM are equal; therefore the remaining gnomon ERO is equal to the remaining *figure* C; and because OR is equal to XS; let RS which is common be added to both; therefore the whole OB is equal to the whole XB: but XB is equal to TE (by 36. 1.) because the side AE is equal to the side EB; therefore also TE is equal to OB; let XS which is common be added; therefore the whole TS is equal to the gnomon ERO: but the gnomon ERO has been demonstrated to be equal to the *figure* C; therefore also TS is equal to the figure C.

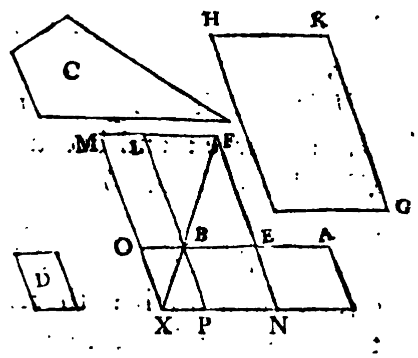
Wherefore the parallelogram TS hath been applied to the given straight line AB equal to the given rectilineal figure C, deficient by the parallelogram figure RS which is similar to D; since (by 24. 6.) RS is similar to OX. Which was to be done.

PROP. XXIX.

To apply a parallelogram, to a given straight line, equal to a given rectilineal figure, exceeding by a parallelogram figure, similar to a given *parallelogram*.

Let

Let AB be the given ſtraight line; and C the given rectilineal figure to which the *parallelogram* applied to AB is required to be equal: and *let* D *be the parallelogram* to which it is required that the exceeding one be ſimilar: it is required to apply a parallelogram to the ſtraight line AB equal to the rectilineal figure C exceeding by a parallelogram figure ſimilar to D. Book VI.



Let AB be cut in halves in the *point* E: and let the parallelogram EL be deſcribed upon EB ſimilar to D and ſimilarly ſituated; and let GH be made equal to both, the *parallelogram* EL and the *figure* C together, and alſo ſimilar to D, and the ſame ſimilarly ſituated (by 25. 6.): therefore GH is ſimilar to EL: but let KH be the *ſide* of like ratio with FL; and KG the *ſide of like ratio* with FE: and becauſe the *parallelogram* GH is greater than EL; therefore the *ſide* KH is greater than FL; and KG than FE: Let FL, FE be produced; and let FLM be equal to KH, and FEN equal to GK; and let the *parallelogram* MN be compleated; therefore MN is equal and ſimilar to GH: but GH is alſo ſimilar to EL; therefore alſo (by 21. 6.) MN is ſimilar to EL: therefore (by 26. 6.) EL is about the ſame diameter with MN: let their diameter FX be drawn; and let the figure be deſcribed.

Wherefore becauſe the *parallelogram* GH is equal to EL and C together (by conſt.); but GH is equal to MN, therefore alſo MN is equal to EL and C together: let EL *which is* common be taken away; therefore the remaining gnomon NOL is equal to the figure C: and becauſe EA is equal to EB; the *parallelogram* AN is alſo (by 36. 1.) equal to NB, that is (by 43. 1.) to LO; let EX *which is* common be added; therefore the whole AX is equal to the gnomon NOL; but the gnomon NOL is equal to the *figure* C; therefore AX is alſo equal to the *figure* C.

Wherefore the parallelogram AX has been applied to the given ſtraight line AB, equal to the given rectilineal figure C; exceeding by the parallelogram figure PO which is ſimilar to D; becauſe (by 24. 6.) EL is ſimilar to OP. Which was to be done.

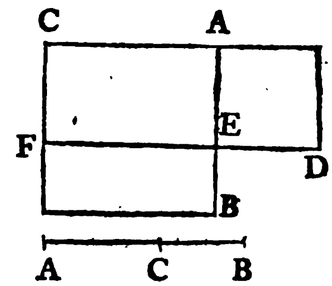
Book VI.

PROP. XXX.

To cut a given finite ſtraight line in extreme and mean ratio.

Let AB be the given finite ſtraight line: it is required to cut the ſtraight line AB in extreme and mean ratio.

For let the ſquare BC be deſcribed upon AB; and (by 29. 6.) let the parallelogram CD be applied to AC, equal to the *ſquare* BC, exceeding by the figure AD ſimilar to BC.



But BC is a ſquare, therefore AD is alſo a ſquare: and becauſe BC is equal to CD, let CE *which is* common be taken away; therefore the remainder BF is equal to the remainder AD; and it is alſo equiangular to it; therefore (by 14. 6.) the ſides which are about the equal angles of the *figures* BF, AD are reciprocally proportional; therefore it is, as FE *is* to ED ſo *is* AE to EB; but (by 34. 1.) FE is equal to AC, that is to AB, and ED to AE: therefore it is, as AB *is* to AE ſo *is* AE to EB: But AB is greater than AE; therefore AE is greater than EB.

Wherefore (by 3. def. 6.) the ſtraight line AB is cut in extreme and mean ratio at the point E; and AE is the greater ſegment of it. Which was to be done.

Otherwise. Let AB be the given ſtraight line; it required to cut the ſtraight line AB in extreme and mean ratio.

For let AB be cut in C (by 11. 2.) ſo that the rectangle *contained* by AB, BC may be equal to the ſquare of AC.

Wherefore becauſe the rectangle *contained* by AB, BC is equal to the ſquare of AC; therefore it is, (by 17. 6.) as AB *is* to AC ſo *is* AC to CB; therefore AB hath been cut in extreme and mean ratio in the point C. Which was to be done.

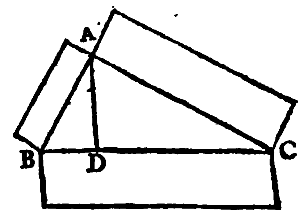
PROP. XXXI.

In right angled triangles, the figure *deſcribed* upon the ſide ſubtending the right angle is equal to the figures ſimilar and ſimilarly deſcribed upon the ſides containing the right angle.

Let

Let ABC be a right angled triangle, having BAC a right angle: I say that the figure described upon BC is equal to the similar and similarly described figures upon BA, AC. Book VI.

Let the perpendicular AD be drawn.



Wherefore because, in the right angled triangle ABC, AD hath been drawn from the right angle at A perpendicular to the base BC; therefore (by 8. 6.) the triangles ABD, ADC, at the perpendicular, are similar to the whole ABC and to one another: and because ABC is similar to ABD; therefore it is, as CB *is* to BA so is AB to BD: and because the three straight lines are proportionals; it is, (by 2. Cor. to 20. 6.) as the first *is* to the third so *is* the figure upon the first to a similar and similarly described figure upon the second; wherefore as CB *is* to BD so *is* the figure upon CB to the similar and similarly described *figure* upon AB. Certainly for the same reason also as BC is to CD so *is* the figure upon CB to the *figure* upon AC: so that it is, (by 24. 5.) as BC is to BD, DC so is the figure upon BC to the figures similar and similarly described upon BA, AC: but BC is equal to BD, DC; therefore the figure upon BC is equal to the figures similar and similarly described upon BA, AC.

Wherefore in right angled triangles, the figure *described* upon the side subtending the right angle is equal to the figures similar and similarly described upon the sides containing the right angle. Which was to be demonstrated.

OTHERWISE. Because similar figures are (by 20. 6.) in the duplicate ratio of the sides of like ratio, therefore the figure upon BC has to the figure upon BA the duplicate ratio of that which BC *has* to BA; but (by Cor. 1. to 20. 6.) the square of BC also has to the square of BA the duplicate ratio of that which BC *has* to BA; therefore also (by 11. 5.) as the figure upon BC *is* to the *figure* upon BA so is the square of BC to the square of BA. Certainly for the same reason, also as the figure upon BC *is* to the figure upon CA so *is* the square of BC to the square of CA: so that also (by 24. 5.) as the figure upon BC is to the figures upon BA, AC so is the square of BC to the squares of BA, AC: but the square of BC is equal (by 47. 1.) to the squares of BA, AC; therefore

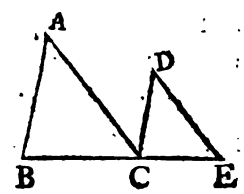
also

Book VI. the figure upon BC is equal to the figures similar and similarly described upon BA, AC. Which was to be demonstrated.

PROP. XXXII.

If two triangles, having two sides proportional to two sides be joined together at one angle, so as that their sides of like ratio may be also parallel; the remaining sides of the triangles will be in a straight line.

Let ABC, DCE be two triangles having the two sides BA, AC proportional to the two sides CD, DE: so that BA *may be* to AC as CD *is* to DE; and AB parallel to CD and AC to DE: I say that BC is in a straight line with CE.



For because AB is parallel to DC, and the straight line AC hath fallen upon them, the alternate angles (by 29. 1.) BAC, ACD are equal to one another: Certainly for the same reason the *angle* CDE is equal to ACD: so that also (by com. not. 1.) the *angle* BAC is equal to CDE: And because there are two triangles ABC, DCE having one angle the *angle* at A equal to one angle the *angle* at D and the sides (by supp.) about the equal angles proportional; as BA *is* to AC so *is* CD to DE; therefore (by 6. 6.) the triangle ABC is equiangular to the triangle DCE: therefore the angle ABC is equal to DCE; and also the *angle* ACD has been demonstrated *to be* equal to BAC: therefore the whole angle ACE is equal to the two angles ABC, BAC; let the common angle ACB be added; therefore the *two* angles ACE, ACB are equal to the *three angles* BAC, ACB, ABC; but (by 32. 1.) the *three angles* BAC, ACB, ABC are equal to two right angles; therefore also the angles ACE, ACB are equal to two right angles: with a certain straight line AC and at the point C in it, two straight lines BC, CE not being towards the same parts, make the adjacent angles ACE, ACB equal to two right angles; therefore (by 14. 1.) BC is in a straight line with CE.

Wherefore if two triangles, having two sides proportional to two sides be joined together at one angle; so as that their sides of like ratio may be parallel: the remaining sides of the triangles will be in a straight line. Which was to be demonstrated.

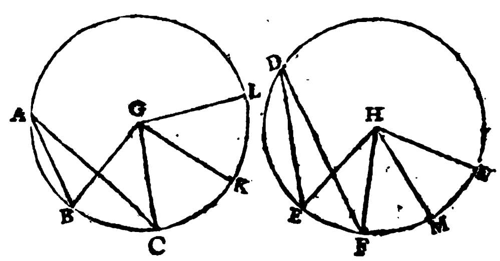
PROP.

PROP. XXXIII.

In equal circles, the angles have the ſame ratio *to one another* as the circumferences on which they ſtand, whether they ſtand at the center or at the circumference. And beſides alſo the ſectors *have the ſame proportion* becauſe they ſtand at the centers.

Let ABC, DEF be equal circles; and let the angles BGC, EHF be at their centers G, H: and the angles BAC, EDF at the circumferences: I ſay that it is, as the circumference BC is to the circumference EF ſo *is* both the angle BGC to the angle EHF and the angle BAC to the angle EDF: and beſides *ſo is* the ſector GBC to the ſector HEF.

Take any number of contiguous *circumferences* (by 1. 4.) CK, KL equal to the circumference BC, and alſo any number *of circumferences which may accidentally happen* FM, MN equal to the circumference EF; and let GK, GL; HM, HN be joined



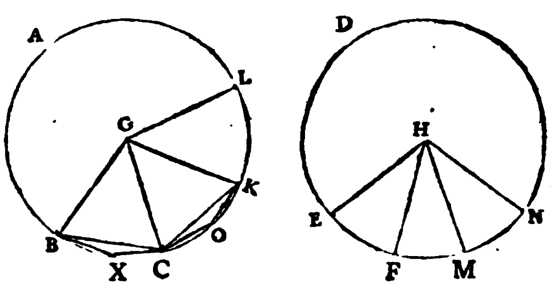
Wherefore becauſe the circumferences BC, CK, KL are equal to one another; alſo (by 27. 3.) the angles BGC, CGK, KGL are equal to one another: whatſoever multiple therefore the circumference BL is of the circumference BC; the ſame multiple alſo is the angle BGL of the angle BGC. Certainly for the ſame reaſon alſo, whatſoever multiple the circumference EN is of the circumference EF; the ſame multiple alſo is the angle EHN of the *angle* EHF: and if the circumference BL be equal to the circumference EN; alſo (by 27. 3.) the angle BGL is equal to the *angle* EHN: and if the circumference BL be greater than the circumference EN; alſo the angle BGL is greater than the angle EHN; and if leſs, leſs: there being four magnitudes; the two circumferences BC, EF; and the two angles BGC, EHF; there have been taken the circumference BL and the angle BGL equimultiples of the circumference BC and of the angle BGC; and the circumference EN and the angle EHN *any other equimultiples which may accidentally happen* of the circumference EF and of the angle EHF; and it has been demonſtrated that if the circumference BL exceed

Book VI. exceed the circumference EN alſo the angle BGL exceeds the angle EHN: and if equal, equal: and if leſs, leſs: it is therefore (by 5. def. 5.) as the circumference BC is to the *circumference* EF ſo is the angle BGC to the *angle* EHF: but as the angle BGC is to the angle EHF ſo is (by 15. 5.) the angle BAC to the angle EDF; for they are (by 20. 3.) the doubles, each of each: therefore as the circumference BC is to the circumference EF ſo *is* both the angle BGC to the angle EHF; and the angle BAC to the angle EDF.

Wherefore in equal circles, the angles have the ſame ratio *to one another* as the circumferences on which they ſtand; whether they ſtand at the centers or at the circumferences. Which was to be demonſtrated.

I ſay alſo, that as the circumference BC *is* to the circumference EF ſo *is* the ſector GBC to the ſector HEF.

For let BC, CK be joined; and the points X, O being taken in the circumferences BC, CK; let BX, XC; CO, OK be joined.



And becauſe the two BG, GC are equal to the two CG, GK; and they contain equal angles; alſo (by 4. 1.) the baſe BC is equal to CK; therefore alſo the triangle BGC is equal to the triangle GCK: and becauſe the circumference BC is equal to the circumference CK; alſo the remaining circumference, which *makes up* the whole circle ABC, is equal to the remaining circumference which *makes up* the ſame circle; ſo that alſo (by 27. 3.) the angle BXC is equal to the angle COK; wherefore (by 11. def. 3.) the ſegment BXC is ſimilar to the ſegment COK; and they are upon equal ſtraight lines BC, CK: but ſimilar ſegments of circles upon equal ſtraight lines (by 24. 3.) are equal to one another: therefore the ſegment BXC is equal to the ſegment COK: but the triangle BGC is alſo equal to the triangle CGK; therefore the whole ſector GBC is equal to the whole ſector GCK. Certainly for the ſame reaſon alſo the ſector GKL is equal to either GKC *or* GCB: therefore the three ſectors GBC, GCK, GKL are equal to one another.

Certainly

Certainly for the ſame reaſon alſo, the ſectors HEF, HFM, HMN are equal to one another; whatſoever multiple therefore the circumference BL is of the circumference BC; the ſector GBL is alſo the ſame multiple of the ſector GBC. Certainly for the ſame reaſon alſo, whatſoever multiple the circumference EN is of the circumference EF; the ſector HEN is alſo the ſame multiple of the ſector HEF: And if the circumference BL be equal to the circumference EN; alſo the ſector GBL is equal to the ſector HEN: and if the circumference BL exceed the circumference EN; the ſector GBL alſo exceeds the ſector HEN; and if *the one* is deficient, *the other* is deficient. There being then four magnitudes, the two circumferences BC, EF; and the two ſectors GBC, HEF; and equimultiples have been taken of the circumference BC, and of the ſector GBC, viz. the circumference BL and the ſector GBL; and equimultiples alſo of the circumference EF, and of the ſector HEF, viz. the circumference EN and the ſector HEN: and it hath been demonſtrated, that if the circumference BL exceed the circumference EN, the ſector GBL alſo exceeds the ſector HEN; and if equal, equal; and if *the one* is deficient *the other* is deficient. Therefore it is (by 5. def. 5.) as the circumference BC is to the circumference EF ſo is the ſector GBC to the ſector HEF.

Cor. And it is manifeſt (by 11. 5.) alſo, that as the ſector is to the ſector ſo is the angle to the angle.

THE END OF THE SIXTH BOOK.

THE CONCLUSION.

AFTER the firſt ſix books of theſe elements are read, it will be very proper for readers of all denominations, to make ſome kind of eſtimate of their improvement in the knowledge of magnitude, by taking a review of the whole. And this will be done with ſome aſtoniſhment by the judicious reader, when he conſiders to what a hight he has raiſed himſelf above the vulgar in his conceptions upon this ſubject; or indeed even above himſelf, when he compares the common perception which he ſet out with, that *Magnitudes which exactly agree together are equal,* with the power of making a rectilineal figure equal to *any* given rectilineal figure, and ſimilar to *any* other which may be given. But as an admiration of the great abilities of the author, and of his judicious arrangement will be the neceſſary conſequence of right notions upon this ſubject; I ſhall decline all encomiums upon theſe, to turn my thoughts to the aſſiſtance of thoſe who may not have reaſon to be ſo well ſatisfied with their improvement; which is likely to be moſt effectually given, by an enumeration of thoſe circumſtances, to which their want of ſucceſs is moſt probably owing; an attention to which, upon ſome future peruſal, may remove every defect. And although I have mentioned them in the courſe of theſe diſſertations yet it may be uſeful as a concluſion to the whole to leave ſuch an impreſſion upon the reader's mind, as a ſummary of the whole is likely to make. The reader therefore is to conſider firſt of all whether he may not have had all this time, even a wrong notion of the very ſubject. The ſubject is magnitude, and not number; nor is it ſufficient for him to be ſatisfied that *things* equal to the ſame are equal to one another; but he muſt convince himſelf, *not give his aſſent*, that magnitudes which are equal to the ſame, are equal to one another; and this is to be done by a particular examination, of as many different kinds of magnitudes

magnitudes as he can easily recollect, especially of such, as are thus compared in these first six books. Next let him examine whether he has a proper notion of a definition, which he must do his endeavour to collect both from its nature and use; still keeping in his mind that every thing here refers to magnitude. And the shewing that any property is consistent with the definition is called a demonstration. But several demonstations ought to be carefully examined by a student in the circumstances of the person to whom I am now writing, entirely with a view to see whether he understands what is meant by a geometrical demonstration; and as we suppose him to have his choice of the first six books; he ought especially to pick out the indirect demonstrations, which are of all others the most elegant and forcible, because they convince you contrary to the representation of the figure: the evidence of such propositions therefore, if its force be perceived at all, comes to the mind purer, as it is not affected by the prejudices which the reasoning upon a particular figure, which seems to represent the consequences, is but to apt to introduce. For such reasonings are often disturbed even by turning the figure upsidedown; and I myself have often puzzled a student sufficiently acute, by such trials as joining AD in the twenty first proposition of the first book, and requiring him to prove that AD and DC together are less than AB and BC together &c. The indirect demonstrations therefore are of all others the properest for a student to exercise himself in; as this will be the means of removing that very great difficulty which learners find in making a ready and proper use of the supposition; as his chief dependence for conviction in the indirect demonstrations, must be upon the supposition. It is true this may be got as I have remarked before, by an examination of the supposition in every proposition, especially such as consist of several circumstances; and varying them, one by one, observing the effect which every variation has upon demonstration; for by such a proceeding the force and use of the different parts of the supposition must be discovered, and indeed hardly by any other means. And in this progress he will be very much assisted by describing his figures will all the variety of positions, which his instruments and the supposition will admit of, until he is certain that he has conquered

quered every prejudice which can arise from reasoning upon a particular figure.

The reader is also carefully to observe the different methods of comparison made use of in the first six books. In the first book, the magnitudes are compared either; as exactly agreeing together; or as being equal to the same magnitude; or as being radius's of the same circle; which are all common and familiar notions. But in the second book, the method of comparison is by the rectangle, which is by no means a common notion, either as to its name or nature; and therefore should be made familiar; as well as perfectly understood. In some part of the third book, the method of comparison is by similar segments, which is also an artificial idea, and ought for the same reason to be made familiar. But the most artificial and therefore the most difficult of all, is that method of comparison founded upon a consideration of quantities as parts and multiples. It will be particularly useful for the student, who has failed in his first attempt to understand this fifth book, after reading carefully what I have said in explanation of the two first definitions, to take care to have the suppositions given him by another; and then carefully to perform every construction himself. And indeed the reader who would take the easiest way of making himself master of this subject, should have the ruler and compasses in his hand, through the first six books, only taking care not to forget, that he is handling a ruler and compasses.

THE END OF THE FIRST VOLUME.